

Distance Recoloring

in collaboration with

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Contents

1 Introduction to Reconfiguration

2 Some Examples

3 Distance Recoloring

4 Concluding Remarks

1 Introduction to Reconfiguration

- Introduction

2 Some Examples

- Example: Token-Set Reconfiguration

- Example: Vertex-Coloring Reconfiguration

3 Distance Recoloring

4 Concluding Remarks

Introduction to Reconfiguration

Reconfiguration Setting

- › A description of what *states* (\equiv *configurations*) are
- › One or more *allowed moves* between states (\equiv *reconfiguration rule(s)*)

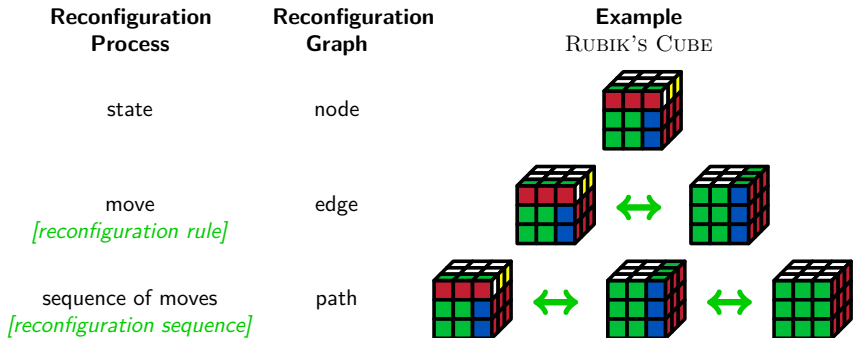


Figure: Reconfiguration Setting

Introduction to Reconfiguration

Reconfiguration vs. Solution Space

For a computational problem \mathcal{P} (e.g., INDEPENDENT SET, DOMINATING SET, VERTEX-COLORING, etc.)

Reconfiguration	Solution Space
States/Configurations	Feasible solutions of \mathcal{P}
Allowed Moves	Slight modifications of a solution <i>without changing its feasibility</i>

Algorithmic Questions

- › **REACHABILITY:** Given two states S and T , is there a sequence of moves that *transforms S into T* ?
- › **SHORTEST TRANSFORMATION:** Given two states S and T and a positive integer ℓ , is there a sequence of moves that *transforms S into T using at most ℓ moves*?
- › **CONNECTIVITY:** Is there a sequence of moves *between any pair of states*?
- › and so on

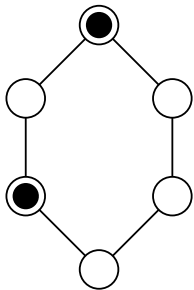
Some Examples

TOKEN RECONFIGURATION

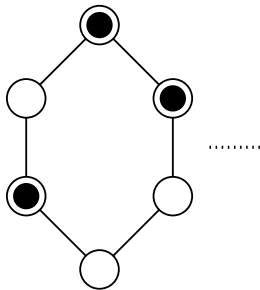
Each vertex has at most one token



Each state is a set of tokens satisfying certain property



Independent Set



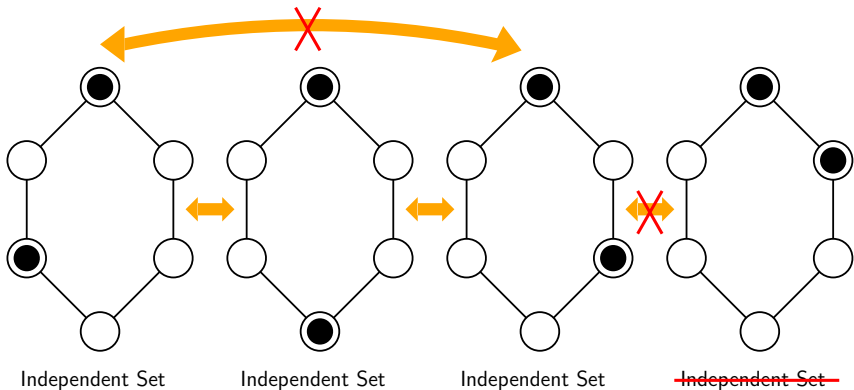
Dominating Set

Some Examples

[Hearn and Demaine 2005]

TOKEN RECONFIGURATION (Example: INDEPENDENT SET RECONFIGURATION)

Token Sliding (TS)



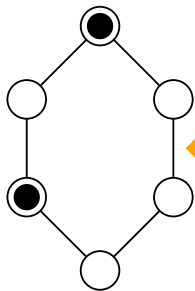
Some Examples

[Hearn and Demaine 2005]

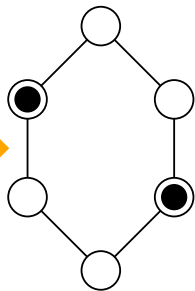
TOKEN RECONFIGURATION (Example: INDEPENDENT SET RECONFIGURATION)

Token Sliding (TS)

PSPACE-complete on planar graphs



Independent Set



Independent Set

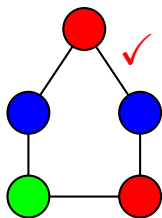
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

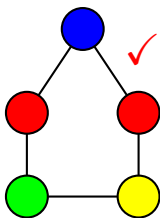
VERTEX-COLORING RECONFIGURATION

Each vertex is colored by one of the k given colors

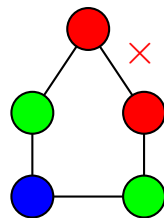
Each state is a k -coloring of all vertices such that no two adjacent vertices share the same color



3-coloring



4-coloring



~~3-coloring~~

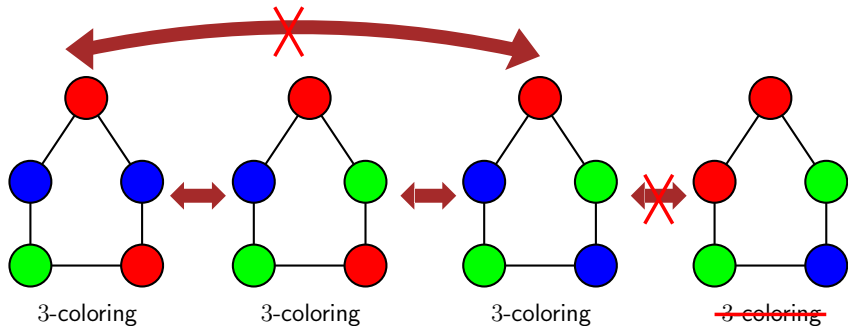
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

Example: $k = 3$



Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

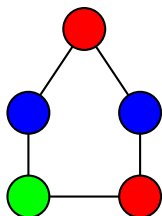
VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

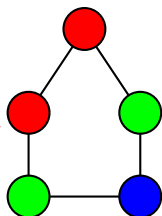
PSPACE-complete on general graphs for $k \geq 4$

Example: $k = 3$

P on general graphs for $k \leq 3$



3-coloring



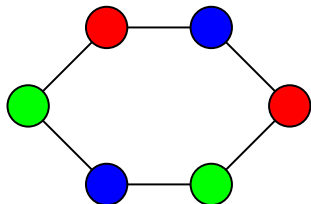
3-coloring

Distance Recoloring

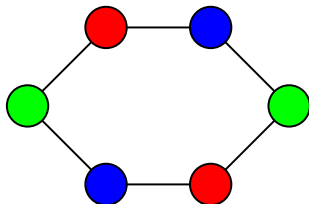
Distance constraints for *vertex colorings*

	k -Coloring	(d, k) -Coloring
Distance between two vertices having the same color	≥ 2	$\geq d + 1$

- ▶ (d, k) -coloring was first studied in [F. Kramer and H. Kramer 1969]
- ▶ Has applications in *frequency assignment problem* (or radio channel assignment) [F. Kramer and H. Kramer 2008]



3-coloring
~~(2, 3)-coloring~~

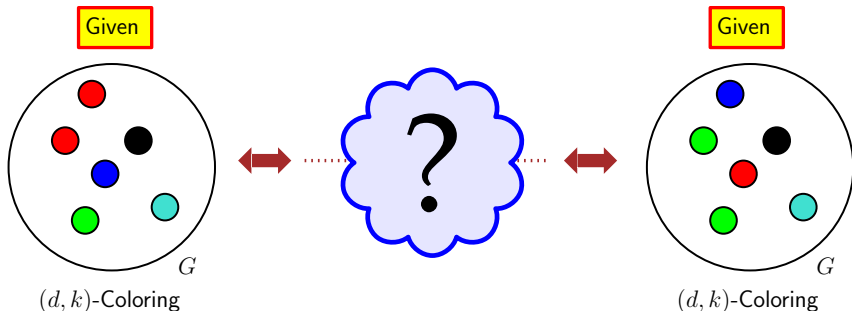


3-coloring
(2, 3)-coloring

Distance Recoloring

(d, k) -COLORING RECONFIGURATION $((d, k)$ -CR)

Reconfiguration Rule: Recoloring a vertex



- The case $d = 1$ (k -COLORING RECONFIGURATION (k -CR)) has been well-studied [Mynhardt and Nasserar 2019]; [Heuvel 2013]
- We focus on the case $d \geq 2$

Distance Recoloring

Graph	k -CR (= $(1, k)$ -CR)	(d, k) -CR ($d \geq 2$)
general	PSPACE-C ($k \geq 3$) [Cereceda, Heuvel, and Johnson 2011]	PSPACE-C ($k = \Theta(nd)$)
planar	PSPACE-C ($4 \leq k \leq 6$) P ($k \geq 7$) [Bonsma and Cereceda 2009]	
bipartite	PSPACE-C ($k \geq 4$) [Bonsma and Cereceda 2009]	
planar \cap bipartite	PSPACE-C ($k = 4$) P ($k \geq 5$) [Bonsma and Cereceda 2009]	
2-degenerate	P [Hatanaka, Ito, and Zhou 2019]	
planar \cap 2-degenerate	P (\subseteq 2-degenerate)	
planar \cap <i>bipartite</i> \cap 2-degenerate	P (\subseteq 2-degenerate)	

Table: Our Results for $d \geq 2$. We provide the status for $d = 1$ for comparison. Here PSPACE-C stands for PSPACE-complete

Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2026)

(d, k) -CR is PSPACE-complete for $d \geq 3$ on *planar, bipartite, and 2-degenerate graphs* and for $d = 2$ on *planar, 2-degenerate graphs*.

Proof Sketch.

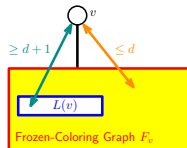
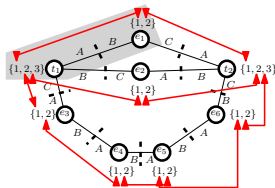
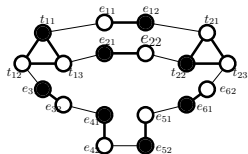
Restricted PSPACE-Complete Variant of ISR under TS
(RESTRICTED SLIDING TOKENS)



LIST $(d, \Omega(d))$ -COLORING RECONFIGURATION



$(d, \Theta(nd))$ -COLORING RECONFIGURATION



Distance Recoloring

First Phase

RESTRICTED SLIDING TOKENS \Rightarrow LIST $(d, \Omega(d))$ -COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

PSPACE-complete on **very restricted instances**

Three types of gadgets

Token triangle

Token edge

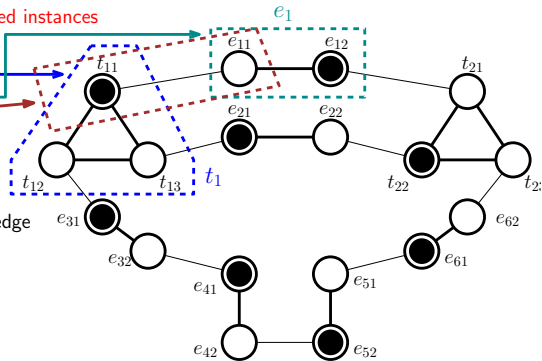
Link edge

Planar

Max degree 3, min degree 2

No two token triangles
are directly joined by a link edge

Each token triangle/token edge
has exactly one token



Distance Recoloring

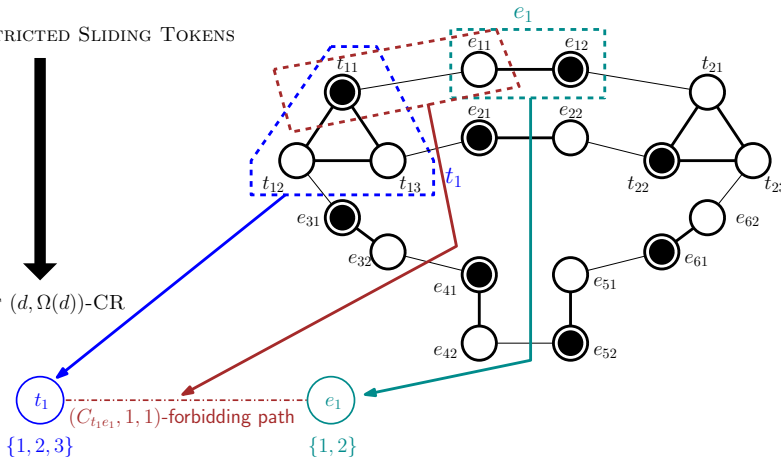
First Phase

RESTRICTED SLIDING TOKENS \Rightarrow LIST $(d, \Omega(d))$ -COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS



LIST $(d, \Omega(d))$ -CR

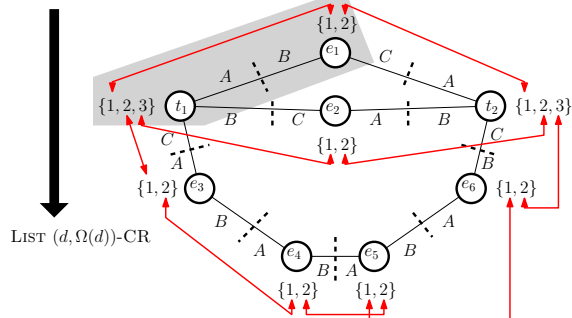


Distance Recoloring

First Phase

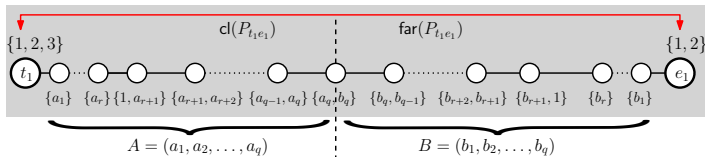
RESTRICTED SLIDING TOKENS \Rightarrow LIST $(d, \Omega(d))$ -COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS



LIST $(d, \Omega(d))$ -CR

Magnification of $(C_{t_1 e_1}, 1, 1)$ -forbidding path $P_{t_1 e_1}$, where $C_{t_1 e_1} = A \cup B$



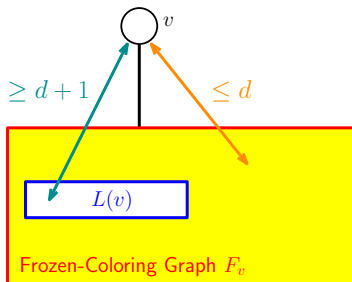
Distance Recoloring

Second Phase

LIST $(d, \Omega(d))$ -COLORING RECONFIGURATION $\Rightarrow (d, \Theta(nd))$ -COLORING RECONFIGURATION

Key Ideas

- 1 List Coloring \equiv Coloring with constraints on which colors can be used for each vertex
- 2 *Frozen-Coloring Graphs*: Pre-colored graphs where *no vertex can be recolored*



Vertices are pre-colored

Containing all possible colors

No vertex can be recolored

Distance Recoloring

Second Phase

LIST $(d, \Omega(d))$ -COLORING RECONFIGURATION $\Rightarrow (d, \Theta(nd))$ -COLORING RECONFIGURATION

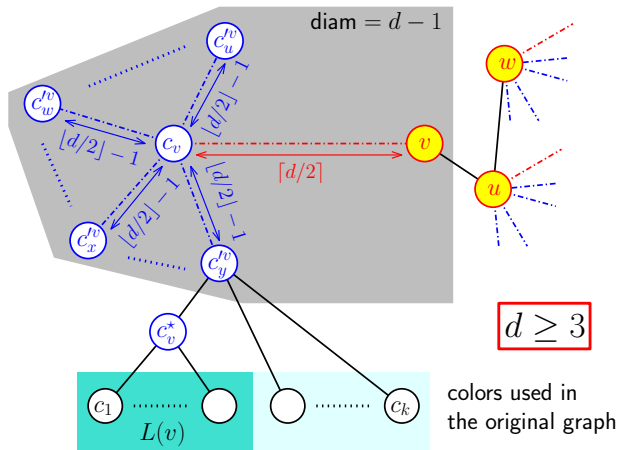


Figure: Construction of the frozen-coloring subgraph F_v for a vertex v . The colors used in this figure are just for illustration of paths.

Concluding Remarks

Take-Home Messages

- 1 *Reconfiguration* studies the “solution space” of a problem
 - » Moving from one solution to another *without violating feasibility*
- 2 Under certain *distance constraints*,
 - » *Reconfiguration problems* can be *hard* for very restricted graph classes

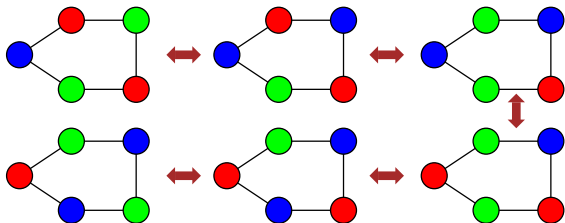
Open Problems

The complexities of the following problem remain *open* for *trees*:

- » (d, k) -CR ($d \geq 2$)



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Came to my talk, you did.
Thank you, I must!

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