On Token Sliding Reconfiguration Graphs of Independent Sets

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Outline

TS$_k$-Graphs: Background and Motivation

TS_k-Graphs: Background and Motivation

For a graph G and fixed $k \geq 1$, $\mathsf{TS}_k(G)$ has:

- **vertices:** size-k independent sets of G (token-sets);
- **edges:** two sets I, J are adjacent iff $J = (I \setminus \{u\}) \cup \{v\}$ for some edge $uv \in E(G)$ and $v \notin I$ (i.e., sliding a token from u to an unoccupied neighbor v).

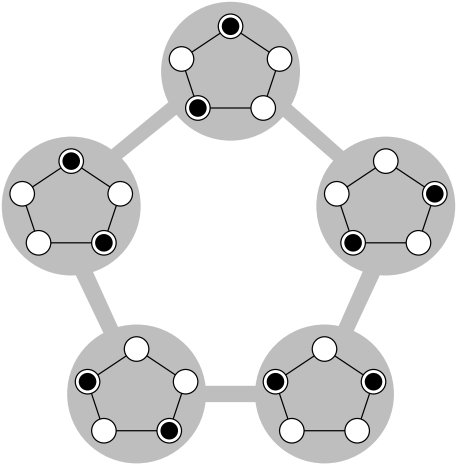


Figure: $\mathsf{TS}_2(C_5)$

TS_k-Graphs: Background and Motivation

General Framework: *Reconfiguration Problems*

- A *growing research area* in computer science which primarily aims to *understand the solution space of a computational problem*
- Given a computational problem Π and two feasible solutions S, T of Π, can we transform S into T by applying a sequence of *small changes* such that all intermediate states are also feasible solutions of Π?
- Under the "reconfiguration setting", $\mathsf{TS}_k(G)$ has been *extremely well-studied* from the *algorithmic perspective* [van den Heuvel 2013]; [Nishimura 2018]; [Bousquet, Mouawad, Nishimura, and Siebertz 2022]
- However, *the structural properties of TS_k-graphs* are still *largely unexplored*

TS_k-Graphs: Technical Challenges

- **Rapidly growing state space:** vertices are all size-k independent sets, so small changes in G may produce many new vertices and adjacencies.
- **Not hereditary:** being a TS_k-graph is not closed under induced subgraphs, even for trees. This prevents a simple forbidden-subgraph characterization as well as any inductive approach
- **Delicate dependence on k:** structural behaviour (e.g., appearance of cycles or realizability of families) can change sharply with small changes in k.
- **Inversion is hard:** building $\mathsf{TS}_k(G)$ from G is easy, but reconstructing a suitable G from a target graph requires careful constructions, labelings, and decompositions.

Paths Are TS_k-Graphs

Paths Are TS_k-Graphs

- We say that a graph H is a *TS_k-graph* if there exists a graph G such that $H \simeq \mathsf{TS}_k(G)$
  - For example, C_5 is a TS_2-graph, since $C_5 \simeq \mathsf{TS}_2(C_5)$
- Since $G \simeq \mathsf{TS}_1(G)$ for any graph G, *the case $k = 1$ is trivial and uninteresting*. From now on, we will always consider *fixed integers $k \geq 2$*
- In this section we illustrate some useful techniques by proving a simple result:

Theorem 1

A path P_n ($n \geq 1$) is a TS_k-graph for any $k \geq 2$

- We present three complementary approaches, each of which *indicates a way of looking at TS_k-graphs*
  1. Characterizing related graphs
  2. Explicit (brute-force) constructions
  3. Characterizing useful graph operations

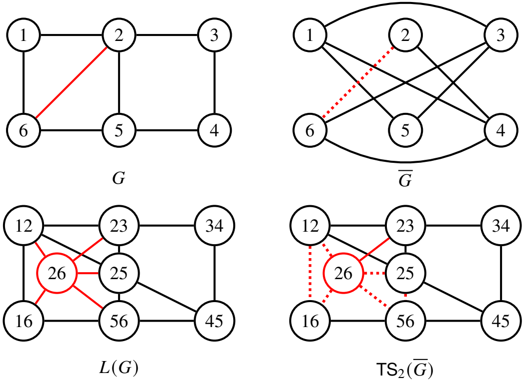
Line Graphs and TS_2-Graphs

Lemma 2

Let $\overline{G}$ and $L(G)$ be respectively the complement and line graph of G

(a) $\mathsf{TS}_2(\overline{G})$ *is a (spanning) subgraph of* $L(G)$

(b) $\mathsf{TS}_2(\overline{G}) \simeq L(G)$ *if and only if* G *is triangle-free*



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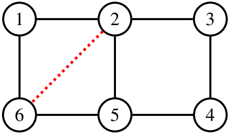
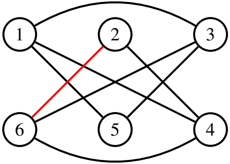
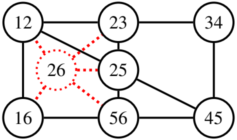


G



$\overline{G}$



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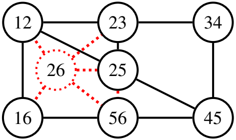

$\mathsf{TS}_2(\overline{G})$

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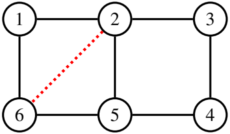

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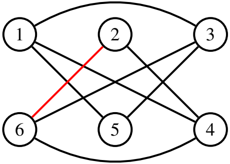

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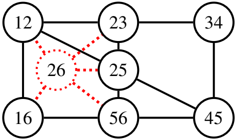

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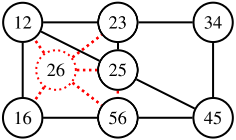

$\mathsf{TS}_2(\overline{G})$

- P_n is triangle-free
- $P_n \simeq L(P_{n+1})$
- Thus, $P_n \simeq \mathsf{TS}_2(\overline{P_{n+1}}) \Rightarrow$ P_n is a TS_2-graph

Labelling Vertices By Stable Sets

Key Idea ($k = 2$)

There exists G such that $P_1 \simeq \mathsf{TS}_2(G)$. For $n \geq 2$, we *update G and label vertices of P_n by size-2 stable sets of G*

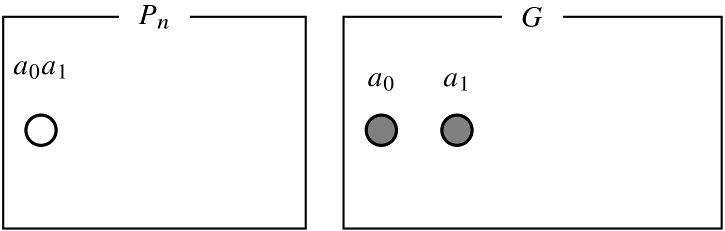

Figure: $P_n \simeq \mathsf{TS}_2(G)$

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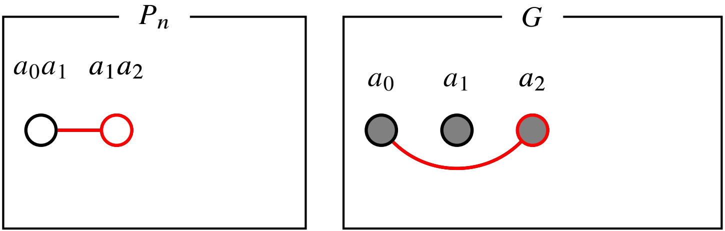

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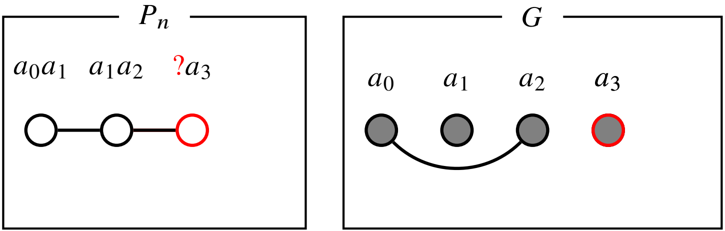

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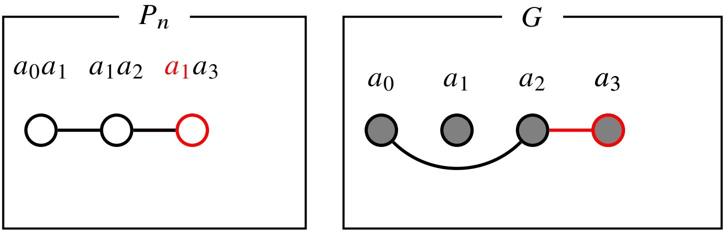

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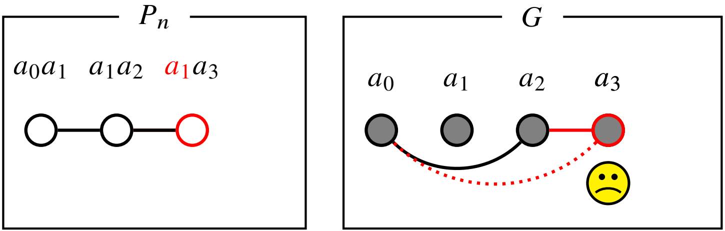

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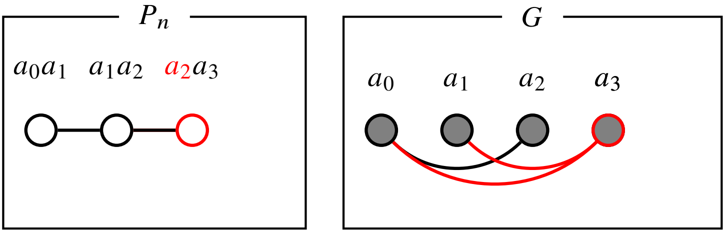

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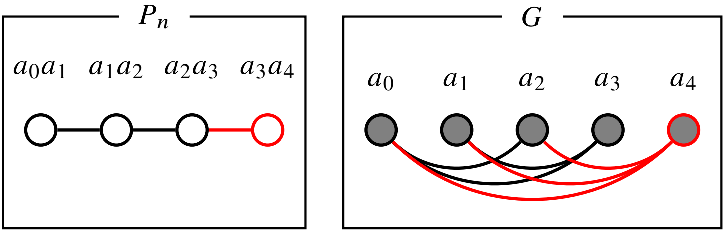


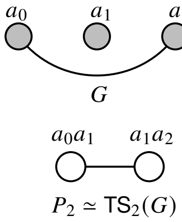
Figure: $P_n \simeq \mathsf{TS}_2(G)$

Gluing Labelled Graphs Together

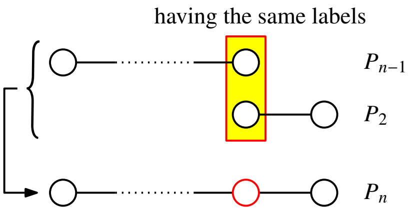
Key Idea ($k = 2$)

(a) There exists a (labelled) graph G such that $P_2 \simeq \mathsf{TS}_2(G)$

(b) With a "good" vertex-labelling (by stable sets of some graph), the TS_2-graph P_n can be constructed by *"gluing" two smaller TS_2-graphs P_{n-1} and P_2*, for any $n \geq 3$



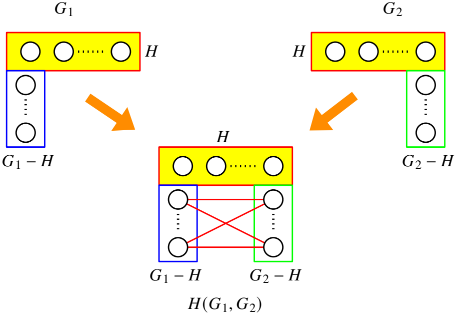
(a)



(b)

Gluing Labelled Graphs Together

- Vertex-labelled graphs G_1 and G_2 are *H-consistent* if *the (possibly empty) intersection of their vertex sets defines the same (possibly empty) common induced subgraph H*
- The *H-join of H-consistent graphs G_1, G_2*, denoted by $H(G_1, G_2)$, is the graph constructed by
  - Putting G_2 on top of G_1 (or vice versa) so that their common induced subgraph H coincide
  - Joining any pair of vertices v, w between $G_1 - H$ and $G_2 - H$, respectively



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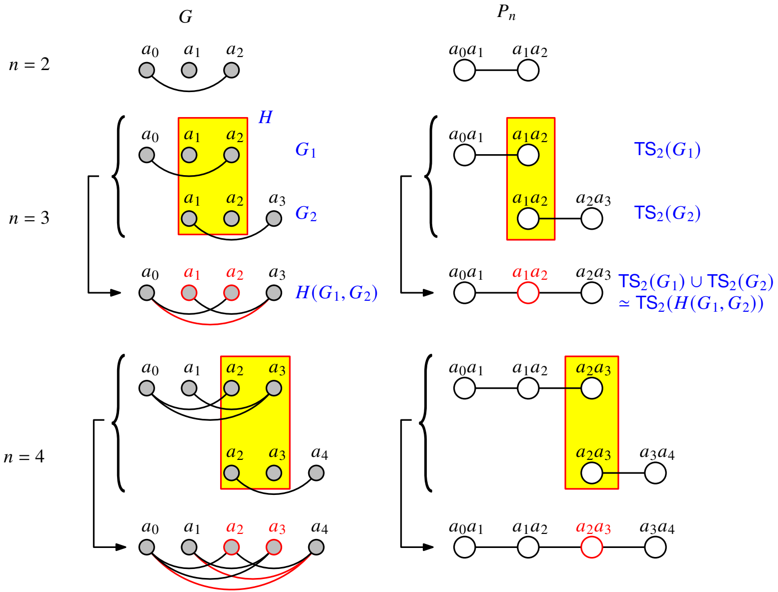


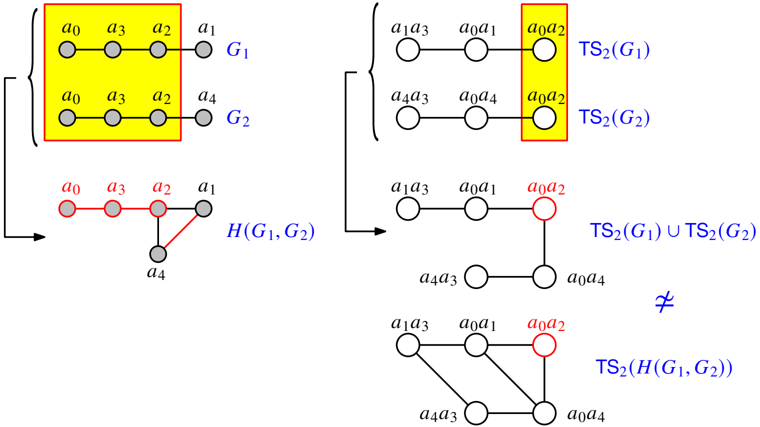
Figure: $P_n \simeq \mathsf{TS}_2(G)$

Gluing Labelled Graphs Together

Remark

There exist H-consistent graphs G_1, G_2 such that

$$\mathsf{TS}_2(G_1) \cup \mathsf{TS}_2(G_2) \not\simeq \mathsf{TS}_2(H(G_1, G_2))$$



Gluing Labelled Graphs Together

In general, the following proposition describes *how to compute the* TS_k*-graph of an* H*-join*

Proposition 3

Let $k \geq 2$ *and let* G_1 *and* G_2 *be two* H*-consistent graphs.* $\mathsf{TS}_k(H(G_1, G_2))$ *is the union of* $\mathsf{TS}_k(G_1)$, $\mathsf{TS}_k(G_2)$ *and for every pair of* k*-element stable sets* S_1 *in* G_1 *and* S_2 *in* G_2 *satisfying*

$$|S_1 \cap V(H)| = |S_2 \cap V(H)| = |S_1 \cap S_2| = k - 1, \tag{1}$$

the edge between S_1 *and* S_2

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- We say that $H(G_1, G_2)$ is *k-crossing free* if there are *no k-element stable sets satisfying condition (1) of Proposition 3*

Corollary 4

Let $k \geq 2$ *and let* G_1 *and* G_2 *be two* H*-consistent graphs.* $H(G_1, G_2)$ *is* k*-crossing free if and only if*

$$\mathsf{TS}_k(H(G_1, G_2)) \simeq \mathsf{TS}_k(G_1) \cup \mathsf{TS}_k(G_2) \tag{2}$$

Final Piece: How to solve for $k \geq 3$?

The following observation *settles the case $k \geq 3$* (note that $\alpha(\overline{P_{n+1}}) = 2$)

- Let $G = \mathsf{TS}_{\alpha(H)}(H)$.
- For $k \geq \alpha(H)$ let H' be H plus $k - \alpha(H)$ isolated vertices.
- Every size-k independent set of H' is a maximum independent set of H together with the added isolates, so moves in $\mathsf{TS}_k(H')$ correspond exactly to moves in $\mathsf{TS}_{\alpha(H)}(H)$.
- Hence $\mathsf{TS}_k(H') \simeq \mathsf{TS}_{\alpha(H)}(H)$.

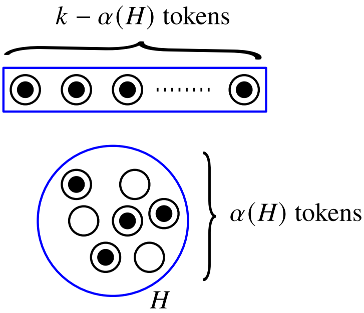

Figure: The graph H'

On TS_k-Graphs

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- If H is an induced subgraph of G then $\mathsf{TS}_k(H)$ is also an induced subgraph of $\mathsf{TS}_k(G)$. The reverse does not hold (e.g., take $H = C_{2k}$ and $G = K_{1,k+1}$)

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- *K_n, $\overline{K_n}$ ($n \geq 1$)* is a TS_k-graph
- *P_n ($n \geq 1$) and C_n ($n \geq 3$)* are TS_k-graphs. On the other hand, *$\overline{P_n}$ ($n \geq 1$) and $\overline{C_n}$ ($n \geq 3$)* are TS_k-graphs for $n \leq 5$ and $n \leq 6$, respectively.

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- $K_{m,n}$ $(1 \leq m \leq n)$ is a TS_k-graph for any $k \geq 2$ if and only if either $m = 1$ and $n \leq k$ or $m = n = 2$ (A similar approach can be used to characterize whether *split graphs* are (not) TS_k-graphs)

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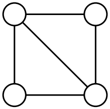
- For any $k \geq 2$,
  - *K_1* is the *smallest graph that is a TS_k-graph*
  - On the other hand, the *diamond* is the *smallest graph that is not a TS_k-graph*

G	H such that $G \simeq \mathsf{TS}_k(H)$ ($k \geq 2$)
 K_1	 k isolated vertices
 diamond	does not exist

On Acyclic TS_k-Graphs

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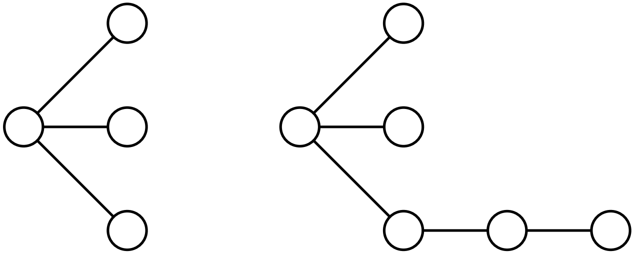
Remark

"Being a TS_k-graph" is *not* hereditary, even when $k = 2$

Using the "labelling vertices by stable sets" approach, we showed that

- $K_{1,3}$ is *not* a TS_2-graph. More generally, *$K_{1,n}$ is a TS_k-graph for some fixed $k \geq 2$ if and only if $n \leq k$*
- Replacing an edge of $K_{1,3}$ by a P_4 results a TS_2-graph



not a TS_2-graph   a TS_2-graph

On Acyclic TS_k-Graphs

Open Question

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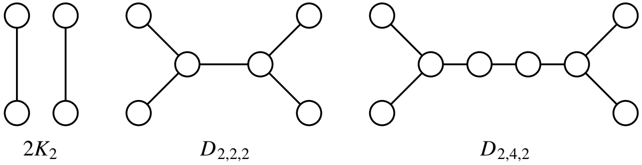
- Given a forest F
  - $\mathsf{TS}_2(F)$ is acyclic if and only if F is $\{2K_2, D_{2,2,2}\}$-free



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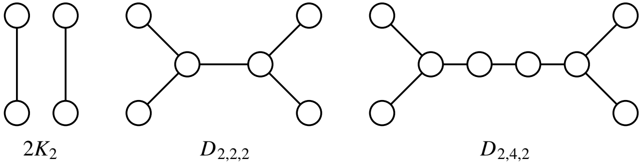
  - $\mathsf{TS}_3(F)$ is acyclic if and only if F is $\{2K_2 + K_1, D_{2,2,2} + K_1, D_{2,4,2}\}$-free



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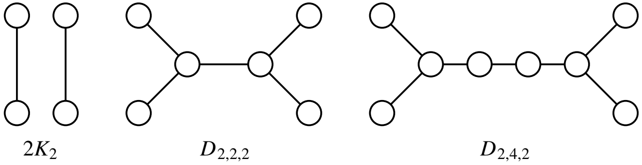
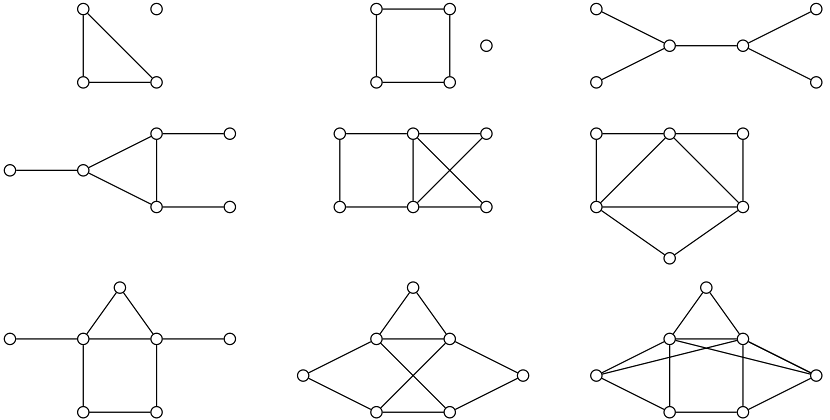
  - $\mathsf{TS}_3(F)$ is acyclic if and only if F is $\{2K_2 + K_1, D_{2,2,2} + K_1, D_{2,4,2}\}$-free
  - **Conjecture:** $\mathsf{TS}_k(F)$ is acyclic if and only if F is $\{2K_2 + (k-2)K_1, D_{2,2,2} + (k-2)K_1, D_{2,4,2} + (k-3)K_1\}$-free, for $k \geq 4$



On Acyclic TS_k-Graphs

- Given a graph G. If G contains either $\overline{C_n}$ *($n \geq 4$)* or *one of the following nine graphs* as an induced subgraph then $\mathsf{TS}_2(G)$ has a cycle
  - The $\overline{C_n}$ ($n \geq 4$) graphs come from the "line graph" approach (Lemma 2)
  - (Most of) The below nine graphs come from a computer program



- **Open Question:** Does the reverse hold? (i.e., Did we miss any graph?)

Thank you for your attention!

References

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