VNU-HUS MAT1206E/3508: Introduction to AI

First-order Predicate Logic
In-class Discussion

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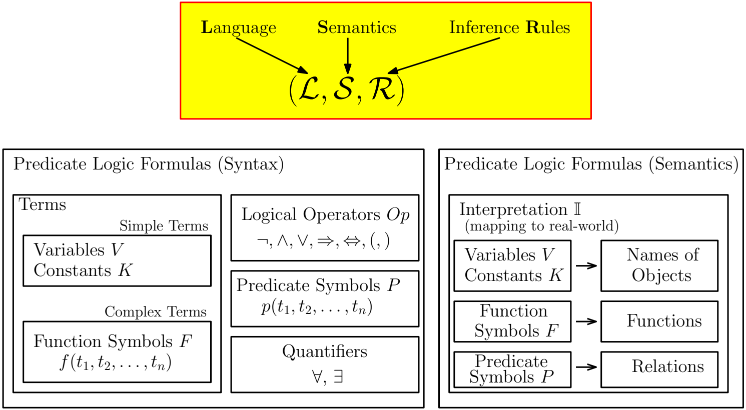
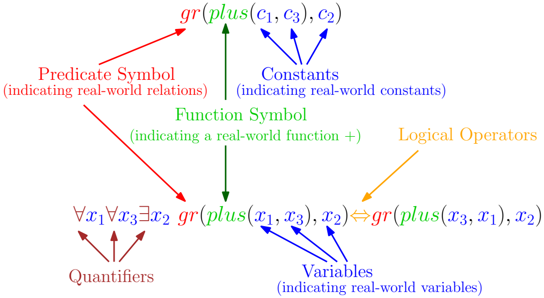
Introduction

- Logic is one of the oldest areas in AI and was the dominant approach from the 1950s through the 1980s (recall symbolic AI).
- Today, machine learning—especially statistical and deep learning methods—drives most practical applications, and logic-based methods are less prominent in mainstream AI.
- Still, logic is crucial for understanding AI's foundations and for applications that require explicit, interpretable, and verifiable reasoning.
  - Task planning for service robots.
  - Verification and decision-making in autonomous driving.
  - Combining symbolic knowledge in predicate logic with sub-symbolic sensor data (e.g., via feature extraction from deep learning) is a promising research direction.
- Building on our earlier discussion of propositional logic, we now introduce predicate logic as a more expressive framework for knowledge representation and reasoning.

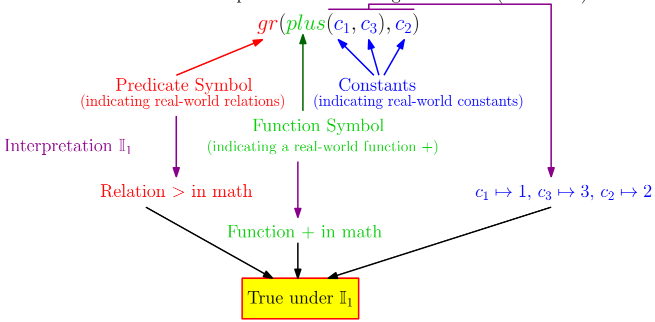
Basics of First-order Predicate Logic
Syntax and Semantics

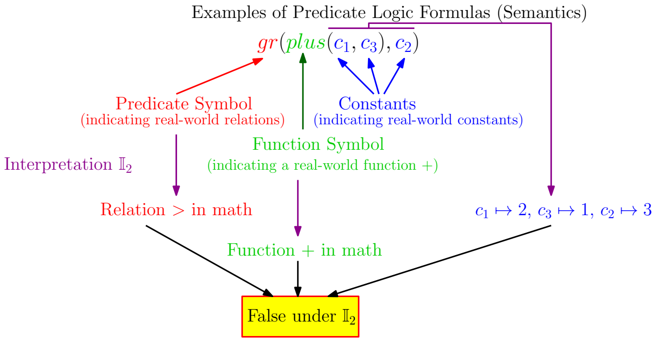


Examples of Predicate Logic Formulas (Syntax)



Examples of Predicate Logic Formulas (Semantics)




Examples of Predicate Logic Formulas (Semantics)
$gr(plus(c_1, c_3), c_2)$
Predicate Symbol
(indicating real-world relations)
Constants
(indicating real-world constants)
Function Symbol
(indicating a real-world function +)
Interpretation $\mathbb{I}_2$
Relation > in math
Function + in math
$c_1 \mapsto 2,\ c_3 \mapsto 1,\ c_2 \mapsto 3$
False under $\mathbb{I}_2$

Exercise 1

Review the following concepts:

(a) Variables, Constants, Function Symbols, Predicate Symbols
(b) (Simple/Complex) Terms
(c) Predicate-Logic Formulas, Literals, Free Variables
(d) CNF, Horn Formulas
(e) Interpretation (Assignment), Truth Value of a Formula
(f) Semantics Equivalence, Satisfiable/Unsatisfiable/Valid Formulas, Models, Semantics Entailment

Exercise 2

(a) What are the input and output of a function symbol? (**Hint:** Think about what a function symbol like *plus*(c_1, c_3) would take as input and what it would return as output)

(b) What are the input and output of a predicate symbol? (**Hint:** Think about what a predicate symbol like *greater*(c_1, c_3) would take as input and what it would return as output)

(c) Are there any differences between a function symbol and a real function in mathematics? Simialrly, are there any differences between a predicate symbol and a real relation? (**Hint:** Think about what a function symbol or a predicate symbol **represents** in real life)

Exercise 3

Using the concepts mentioned in the previous exercises, explain the family tree example in [Ertel 2025], Example 3.2, p. 45. You may start by answering the following questions:

(a) Which concepts from the above list are used in the example?

(b) What are the “meanings” of the predicate symbols in the example? Give examples of interpretations (assignments) which make the predicates true/false.

(c) What is the knowledge base KB formed in the example? Can you give an example of a query Q which we hope to derive from KB (other than the given query in the example)?

- To be able to *compare terms*, *equality* is a very important relation in predicate logic
- The *equality of terms in mathematics* is an *equivalence relation*, meaning it is *reflexive*, *symmetric* and *transitive*
- We define a *predicate "="* using infix notation as is customary in mathematics (that is, instead of writing "*eq*(x, y)", we write "$x = y$")

Equality Axioms

$\forall x$	$x = x$	(reflexive)
$\forall x \, \forall y$	$x = y \Rightarrow y = x$	(symmetry)
$\forall x \, \forall y \, \forall z$	$x = y \wedge y = z \Rightarrow x = z$	(transitivity)

To guarantee the uniqueness of functions, we additionally require that for any function symbol f and any predicate symbol p

$\forall x \, \forall y$	$x = y \Rightarrow f(x) = f(y)$	(substitution axiom)
$\forall x \, \forall y$	$x = y \Rightarrow p(x) \Leftrightarrow p(y)$	(substitution axiom)

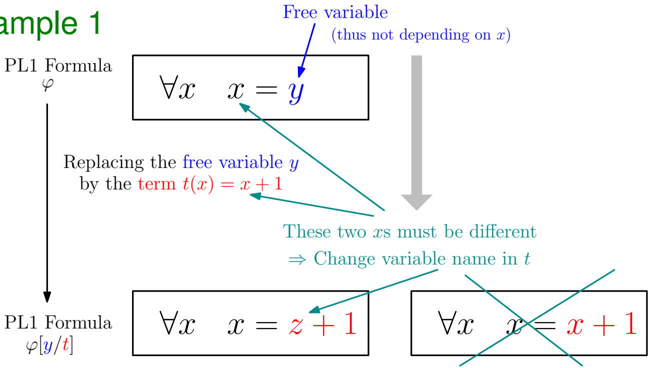
Exercise 4

Can you define the predicate “$<$” in a similar way as the equality “=”?

Replacing a variable by a term

We write $\varphi[x/t]$ for the formula that results when we *replace every free occurrence of the variable x in φ with the term t*. Thereby we *do not allow any variables in the term t that are quantified in φ*. In those cases *variables must be renamed* to ensure this

Example 1



Substitution

A *substitution* σ is *a map from variables to terms*.

- ϵ: the empty substitution
- $\sigma : x_1/t_1, x_2/t_2, \ldots, x_n/t_n$: a substitution that maps each variable x_i to the corresponding term t_i
- Also write $\sigma = \{x_1/t_1, x_2/t_2, \ldots, x_n/t_n\}$

- *Apply σ to a term t*, denoted by $\sigma(t)$ or $t[x_1/t_1, \ldots, x_n/t_n]$ to indicate $\sigma(t)$, means *simultaneously replacing every occurrence of each x_i in t by t_i*
- As before, *we do not allow any variables in the term t_i that are quantified in t*. In those cases *variables must be renamed* to ensure this

Negating Quantified Formulas

$$\neg\forall x\, \varphi \equiv \exists x\, \neg\varphi$$
$$\neg\exists x\, \varphi \equiv \forall x\, \neg\varphi$$

Prenex Normal Form (PNF)

A formula is in *prenex normal form* if it is of the form

$$Q_1x_1Q_2x_2\dots Q_nx_n\psi$$

where each Q_i is a quantifier ($\forall$ or $\exists$), x_i are variables, and ψ is a quantifier-free formula

Theorem 1

For every formula φ there is an equivalent formula φ' in prenex normal form

Example 2

- Let $\varphi = \forall x\,(P(x) \to \exists y\, Q(x, y))$.
- φ is not in prenex normal form **[Why?]**
- An equivalent prenex form: $\varphi' = \forall x\,\exists y\,(P(x) \to Q(x, y))$. **[Verify!]**

Exercise 5

Review the following algorithms:

(a) Transforming a formula into an equivalent formula in Prenex Normal Form (PNF)

(b) Skolemization:

- Eliminate existential quantifiers by introducing Skolem functions
- If the original formula is true, then the resulting formula is also true. The converse is not always true

To have a better understanding, try to apply these algorithms to some predicate-logic formulas.

- Our main goal is to introduce the *resolution calculus* for predicate logic
- We shall first begin with a simpler calculus formed by two inference rules: Modus Ponens and ∀-Elimination
  - *Modus Ponens (MP)*

$$\frac{A, A \Rightarrow B}{B}$$

  - *∀-Elimimation (∀-E)*

$$\frac{\forall x\ A}{A[x/t]}$$

Note: t is a ground term that *contains no variables*

KB

$$female(karen) \quad female(anne) \quad female(mary)$$

$$female(eve) \quad female(isabelle)$$

$$child(oscar, karen, frank) \quad child(mary, karen, frank)$$

$$child(eve, anne, oscar) \quad child(henry, anne, oscar)$$

$$child(isabelle, anne, oscar) \quad child(clyde, mary, oscarb)$$

$$\forall x \, \forall y \, \forall z \; child(x, y, z) \Rightarrow child(x, z, y)$$

$$\forall x \, \forall y \; descendant(x, y) \Leftrightarrow (\exists z \; child(x, y, z) \lor (\exists u \, \exists v \; child(x, u, v) \land descendant(u, y)))$$

$?$

Q

$$child(eve, oscar, anne)$$

KB

$$female(karen) \quad female(anne) \quad female(mary)$$

$$female(eve) \quad female(isabelle)$$

$$child(oscar, karen, frank) \quad child(mary, karen, frank)$$

$$child(eve, anne, oscar) \quad child(henry, anne, oscar)$$

$$child(isabelle, anne, oscar) \quad child(clyde, mary, oscarb)$$

$$\forall x \forall y \forall z \; child(x, y, z) \Rightarrow child(x, z, y)$$

$$\forall x \forall y \; descendant(x, y) \Leftrightarrow (\exists z \; child(x, y, z) \vee (\exists u \exists v \; child(x, u, v) \wedge descendant(u, y)))$$

?

Q

$$child(eve, oscar, anne)$$

Step	Proved by
1. $child(eve, anne, oscar)$	KB
2. $\forall x \forall y \forall z \; child(x, y, z) \Rightarrow child(x, z, y)$	KB
3. $child(eve, anne, oscar) \Rightarrow child(eve, oscar, anne)$	$\forall$-E for 2: x/eve, $y/anne$, $z/oscar$
4. $child(eve, oscar, anne)$	MP for 1 and 3

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Exercise 6

In the textbook, the author claimed in page 52 that "The calculus consisting of the two given inference rules is not complete." where here "the two given inference rules" are Modus Ponens and ∀-Elimination.

To prove that the claim is indeed correct, give an example of a query Q that cannot be derived from the above example KB using only these two inference rules. (That is, $KB \models Q$ but $KB \not\vdash Q$ using only Modus Ponens and ∀-Elimination)

(**Hint:** Think about what appears in the knowledge base but not in the inference rules)

Theorem 2 (Gödel's completeness theorem [Godel 1931])

First-order predicate logic is complete. That is, there is a calculus with which every proposition that is a consequence of a knowledge base KB can be proved. If $KB \models \varphi$, then it holds that $KB \vdash \varphi$.

Note

Indeed, as we will see later, there is a calculus (e.g., resolution calculus) with which only true propositions can be proved. That is, if $KB \vdash \varphi$ holds, then $KB \models \varphi$.

Triggered by the resolution calculus (introduced in 1965), around 1970s, many scientists believed that one could formulate almost every task of knowledge representation and reasoning in PL1 and then solve it with an automated prover.

Figure: The Universal Logic Machine (from [Ertel 2025], Figure 3.3, p. 53)

Recall: Resolution Rule in Propositional Logic

$$\frac{A \vee B, \neg B \vee C}{A \vee C} \quad \text{or} \quad \frac{A \vee B, B \Rightarrow C}{A \vee C}$$

Recall: Resolution Proof in Propositional Logic

- **Input:** Knowledge base KB
- **Goal:** Decide whether $KB \models Q$
- **Method:** Add $\neg Q$ to the knowledge base. If the empty clause can be derived, conclude $KB \models Q$. If there is no more resolvable pair of clauses (and the empty clause is not derived), conclude $KB \not\models Q$

Generalized Resolution Rule for PL1

The resolution rule for two clauses in CNF reads

$$\frac{A_1 \vee \dots A_m \vee B, \neg B' \vee C_1 \vee \dots \vee C_n \quad \sigma(B) = \sigma(B')}{\sigma(A_1) \vee \dots \vee \sigma(A_m) \vee \sigma(C_1) \vee \dots \vee \sigma(C_n)},$$

where *σ is the MGU of B and B'*

What does this definition mean?

- Premise 1: $A_1 \vee \dots A_m \vee B$
- Premise 2: $\neg B' \vee C_1 \vee \dots \vee C_n$
- $\sigma(B) = \sigma(B')$ means that B and B' are matched by applying the MGU σ **[What is σ? How to find it?]**
- Apply σ for every literal in each premise
- Now, Premise 1 becomes $\sigma(A_1) \vee \dots \sigma(A_m) \vee \sigma(B)$ and Premise 2 becomes $\neg\sigma(B') \vee \sigma(C_1) \vee \dots \vee \sigma(C_n)$
- The usual resolution rule can now be applied

Exercise 7

Explain how to obtain the resolution rule in propositional logic from the generalized resolution rule for PL1

For Completeness, we need an additional inference rule

Factorization Rule for PL1

Factorization of a clause is accomplished by

$$\frac{A_1 \vee A_2 \vee \cdots \vee A_n, \quad \sigma(A_1) = \sigma(A_2)}{\sigma(A_2) \vee \cdots \vee \sigma(A_n)},$$

where σ is the MGU of A_1 and A_2

What does this definition mean?

- Premise: $A_1 \vee A_2 \vee \cdots \vee A_n$
- $\sigma(A_1) = \sigma(A_2)$ means that A_1 and A_2 are matched after their MGU σ is applied
- Apply σ for every literal in the premise
- Now, the Premise becomes $\sigma(A_1) \vee \sigma(A_2) \vee \ldots \sigma(A_n)$
- As $p \vee p \equiv p$, the Conclusion $\sigma(A_2) \vee \ldots \sigma(A_n)$ is derived
- Intuitively, after σ is applied, as $\sigma(A_1) = \sigma(A_2)$, one of these literals becomes “redundant” and can be “removed”

$$female(karen) \quad female(anne) \quad female(mary)$$

$$female(eve) \quad female(isabelle)$$

$$child(oscar, karen, frank) \quad child(mary, karen, frank)$$

$$child(eve, anne, oscar) \quad child(henry, anne, oscar)$$

$$child(isabelle, anne, oscar) \quad child(clyde, mary, oscarb)$$

$$\forall x\, \forall y\, \forall z\; child(x, y, z) \Rightarrow child(x, z, y)$$

$$\forall x\, \forall y\; descendant(x, y) \Leftrightarrow (\exists z\; child(x, y, z) \vee (\exists u\, \exists v\; child(x, u, v) \wedge descendant(u, y)))$$

KB $\xrightarrow{?}$ Q

$$child(eve, oscar, anne)$$

$\sigma : x/eve, y/anne, z/oscar$

$\neg B'$: $\neg child(x, y, z) \vee child(x, z, y)$ $\equiv$

B: $child(eve, anne, oscar)$

$$\text{Gen. Res.} \; \frac{child(eve, anne, oscar),\; child(x, y, z) \Rightarrow child(x, z, y) \qquad \sigma(B) = \sigma(B')}{child(eve, oscar, anne)}$$

$\uparrow$ $\sigma(child(x, z, y))$

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With the above generalized rules, a resolution proof for the family example would be:

Step	Proved by
1. *child*(*eve*, *anne*, *oscar*)	KB
2. ¬*child*$(x, y, z) \vee$ *child*(x, z, y)	KB
3. ¬*child*(*eve*, *oscar*, *anne*)	$\neg Q$
4. *child*(*eve*, *oscar*, *anne*)	GenRes$(1, 2)$
5. $()$	GenRes$(3, 4)$

Exercise 8

Write a resolution proof for the query $Q = \exists y\, \textit{knows}(\textit{henry}, y)$ with respect to the knowledge base $KB = \forall x\, \textit{knows}(x, \textit{mother}(x))$, where $\textit{mother}(x)$ is a function that returns the mother of x, $\textit{knows}(x, y)$ means "x knows y", and *henry* is a constant representing a person named Henry.

Exercise 9

Use the resolution calculus to prove that the formula $R = \forall x\, (\neg \textit{shaves}(\textit{baber}, x) \vee \neg \textit{shaves}(x, x)) \wedge (\textit{shaves}(x, x) \vee \textit{shaves}(\textit{baber}, x))$ (a.k.a the Russell's paradox) is unsatisfiable, where $\textit{shaves}(x, y)$ means "x shaves y" and *baber* is a constant representing a barber. (**Hint:** Remeber that you can also use the Factorization rule along with the Generalized Resolution rule.)

Note

- The *search for a proof* can be *very frustrating in practice*
  - Even when $KB \wedge \neg Q$ has only a few clauses, *every resolution step generates a new clause* which increases the number of possible resolution steps in the next iteration

Exercise 10

Do your own research on some *strategies* that can be used to carry out a resolution proof more efficiently.

- Unit Resolution
- Set of Support (SOS) Strategy
- Input Resolution
- Pure Literal Rule
- Subsumption

Exercise 11

Equality is a particularly problematic source of search-space explosion. When the equality axioms are included in the knowledge base, they can generate new clauses and equations that in turn trigger further applications of the equality axioms, potentially causing unbounded growth.

Because of this, special inference rules for equality like *demodulation* and more generally *paramodulation* have been developed which get by without explicit equality axioms and, in particular, reduce the search space. Do your own research on these two inference rules and explain how they work.

Automated Theorem Provers

- An *automated theorem prover* is a software tool that uses logical reasoning to automatically prove or disprove mathematical theorems or logical statements
- It takes a set of axioms and a conjecture as input and attempts to derive the conjecture from the axioms using formal logic rules
- Automated theorem provers are used in various fields, including mathematics, computer science, and artificial intelligence, to assist in formal verification, program analysis, and knowledge representation
- We demonstrate how to use an automated theorem prover called *E* to solve some problems in first-order predicate logic

Automated Theorem Provers

Figure: E Theorem Prover

https://wwwlehre.dhbw-stuttgart.de/~sschulz/E/E.html

Automated Theorem Provers

Definition

A structure $(M, \cdot)$ consisting of a set M with a two-place inner operation "·" is called a semigroup if the law of associativity

$$\forall x\, \forall y\, \forall z\, (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

holds. An element $e \in M$ is called *left-neutral* (*right-neutral*) if $\forall x\, e \cdot x = x$ ($\forall x\, x \cdot e = x$).

Theorem 3

If a semigroup has a left-neutral element e_l and a right-neutral element e_r, then $e_1 = e_r$.

Goal

Prove Theorem 3

Proof of Theorem 3.

For every $x \in M$, it holds that

$$e_l \cdot x = x \tag{1}$$

$$x \cdot e_r = x \tag{2}$$

Replacing $x = e_r$ in Eq. (1) and $x = e_l$ in Eq. (2), we have

$$e_l \cdot e_r = e_r \tag{3}$$

$$e_l \cdot e_r = e_l \tag{4}$$

Combining Eq. (3) and Eq. (4), we have $e_l = e_l \cdot e_r = e_r$. □

The function $m(x, y) = x \cdot y$.

- Negated query $(\neg e_l = e_r)_1$
- Knowledge Base KB
  - Definitions of semi-groups and left-/right- neutrals.

$$(m(m(x, y), z) = m(x, m(y, z)))_2 \quad \text{associativity}$$
$$(m(e_l, x) = x)_3 \quad \text{left-neutral}$$
$$(m(x, e_r) = x)_4 \quad \text{right-neutral}$$

  - Equality axioms (for comparing terms)

$$(x = x)_5 \quad \text{reflexive}$$
$$(\neg x = y \vee y = x)_6 \quad \text{symmetry}$$
$$(\neg x = y \vee \neg y = z \vee x = z)_7 \quad \text{transitivity}$$
$$(\neg x = y \vee m(x, z) = m(y, z))_8 \quad \text{substitution for } m$$
$$(\neg x = y \vee m(z, x) = m(z, y))_9 \quad \text{substitution for } m$$

A resoution proof may be as follows.

Step	Proved by
10. $x = m(e_l, x)$	Res$(3, 6, x_6/m(e_l, x_3), y_6/x_3)$
11. $\neg m(e_l, x) = z \vee x = z$	Res$(7, 10, x_7/x_{10}, y_7/m(e_l, x_{10}))$
12. $e_r = e_l$	Res$(4, 11, x_4/e_l, x_{11}/e_r, z_{11}/e_l)$
13. $()$	Res$(1, 12, \emptyset)$

For example, Res$(3, 6, x_6/m(e_l, x_3), y_6/x_3)$ means that in the resolution of clause 3 with clause 6, the x in clause 6 is replaced by $m(e_l, x)$ where the variable x is from clause 3 and y from clause 6 is replaced by x from clause 3.

Transformation in clause normal form language LOP (The syntax of LOP represents an extension of the PROLOG syntax for non Horn clauses.)

$$(\neg A_1 \vee \cdots \vee \neg A_m \vee B_1 \vee \cdots \vee B_n) \mapsto \mathtt{B}_1;\ldots;\mathtt{B}_n\mathtt{<-}\mathtt{A}_1,\ldots,\mathtt{A}_m$$

An input file for E

halbgr1.lop

```
<- eq(el,er). % query
eq( m(m(X,Y),Z), m(X,m(Y,Z)) ). % associativity of m
eq( m(el,X), X ). % left-neutral element of m
eq( m(X,er), X ). % right-neutral element of m
eq(X,X). % equality: reflexivity
eq(Y,X) <- eq(X,Y). % equality: symmetry
eq(X,Z) <- eq(X,Y), eq(Y,Z). % equality: transitivity
eq( m(X,Z), m(Y,Z) ) <- eq(X,Y). % equality: substitution in m
eq( m(Z,X), m(Z,Y) ) <- eq(X,Y). % equality: substitution in m
```

Run `eprover --proof-object halbgr1.lop | epclextract`

```
0 :  :  (eq(X1,X2)| (eq(X2,X1))) :  initial("halbgr1.lop", at_line_6_column_1)
1 :  :  (eq(X1,X2)|( (eq(X1,X3))| (eq(X3,X2)))) :  initial("halbgr1.lop", at_line_7_column_1)
2 :  :  eq(m(el,X1),X1) :  initial("halbgr1.lop", at_line_3_column_1)
3 :  :   (eq(el,er)) :  initial("halbgr1.lop", at_line_1_column_1)
4 :  :  eq(m(X1,er),X1) :  initial("halbgr1.lop", at_line_4_column_1)
5 :  :  (eq(X1,X2)| (eq(X2,X1))) :  fof_simplification(0)
6 :  :  (eq(X1,X2)|( (eq(X1,X3))| (eq(X3,X2)))) :  fof_simplification(1)
7 :  :  [++eq(X1,X2),--eq(X2,X1)] :  5
8 :  :  [++eq(m(el,X1),X1)] :  2
9 :  :   (eq(el,er)) :  fof_simplification(3)
10 :  :  [++eq(X1,X2),--eq(X1,X3),--eq(X3,X2)] :  6
11 :  :  [++eq(X1,m(el,X1))] :  pm(7,8)
12 :  :  [--eq(el,er)] :  9
13 :  :  [++eq(X1,X2),--eq(m(el,X1),X2)] :  pm(10,11)
14 :  :  [++eq(m(X1,er),X1)] :  4
15 :  :  [--eq(er,el)] :  pm(12,7)
16 :  :  [] :  sr(pm(13,14),15) :  'proof'
```

Note:

- Due to the difference between versions of E, the output may be different from the above figure.
- Positive literals are identified by ++ and negative literals by --.
- `pm(a, b)` stands for a resolution step between clause `a` and clause `b`. (`pm` means Paramodulation.)

We'll walk through the prover's steps, focusing on intuition.

Starting Rules (Lines 0–4)

- **Line 0 (Symmetry)**: If $x_2 = x_1$, then $x_1 = x_2$.

$$eq(x_1, x_2) \vee \neg eq(x_2, x_1)$$

- **Line 1 (Transitivity)**: If $x_1 = x_3$ and $x_3 = x_2$, then $x_1 = x_2$.

$$eq(x_1, x_2) \vee \neg eq(x_1, x_3) \vee \neg eq(x_3, x_2)$$

- **Line 2 (Left Identity)**: For any x, $e_l \cdot x = x$.

$$eq(m(e_l, x_1), x_1)$$

- **Line 3 (Negated Conjecture)**: Assume $e_l \neq e_r$.

$$\neg eq(e_l, e_r)$$

- **Line 4 (Right Identity)**: For any x, $x \cdot e_r = x$.

$$eq(m(x_1, e_r), x_1)$$

Simplifying the Rules (Lines 5–10)

- **Lines 5–6**: The prover rewrites Lines 0 and 1 into a standard form for processing. Think of it as organizing the rules.
- **Lines 7–10**: Converts axioms into a logical format:
  - Line 8: $[++eq(m(e_l, x_1), x_1)]$ — Restates $e_l \cdot x = x$.
  - Line 9: $\neg eq(e_l, e_r)$ — Restates $e_l \neq e_r$.
  - Line 10: Transitivity in logical form.

First Big Step (Line 11)

- The prover uses symmetry (Line 7) and left identity (Line 8) to derive:

$$[++eq(x_1, m(e_l, x_1))]$$

- **Intuition**: If $e_l \cdot x = x$, then $x = e_l \cdot x$. Like saying: "If $1 \cdot x = x$, then $x = 1 \cdot x$." A small logical flip to rewrite equations.

Automated Theorem Provers

Proof By E Theorem Prover

Combining Transitivity (Line 13)

- Combines transitivity (Line 10) with Line 11 to derive:

$$[+ + eq(x_1, x_2), - - eq(m(e_l, x_1), x_2)]$$

- **Intuition**: Either $x = y$ or $e_l \cdot x \neq y$. If $e_l \cdot x$ isn't equal to y, then x can't be y. Sets up a rule for testing equalities.

Automated Theorem Provers
Proof By E Theorem Prover

Using the Right Identity (Line 14 and Line 16)

- **Line 14**: $[++eq(m(x_1, e_r), x_1)]$ — Restates $x \cdot e_r = x$.
- In Line 16, combines Line 13 with Line 14:

$$eq(x_1, m(x_1, e_r)) \vee \neg eq(m(e_l, x_1), m(x_1, e_r))$$

- Since $m(x_1, e_r) = x_1$, simplifies to:

$$eq(x_1, x_1) \vee \neg eq(m(e_l, x_1), x_1)$$

- Since $x_1 = x_1$ is true, this becomes:

$$\neg eq(m(e_l, x_1), x_1)$$

- **Problem**: This says $e_l \cdot x \neq x$, contradicting Line 8 ($e_l \cdot x = x$). Like saying $1 \cdot x \neq x$, which can't be true.

Tying It to the Goal (Line 15 and Line 16)

- **Line 15**: $[--eq(e_r, e_l)]$ — Says $e_r \neq e_l$, from $e_l \neq e_r$ via symmetry.
- In Line 16, set $x_1 = e_r$ in $\neg eq(m(e_l, x_1), x_1)$:

$$\neg eq(m(e_l, e_r), e_r)$$

- **Why it matters**: Line 8 says $e_l \cdot e_r = e_r$, so $e_l \cdot e_r \neq e_r$ is false. If $e_l = e_r$, then $e_r = e_l$, contradicting $e_r \neq e_l$.
- Result: Empty clause ([]), meaning "this math doesn't add up." Thus, $e_l \neq e_r$ is impossible, so $e_l = e_r$.

Exercise 12

Use the E theorem prover to solve [Ertel 2025], Exercise 3.9, p. 64. Prepare the LOP input file and run E (for example: `eprover -proof-object <file> | epclextract`). Collect the prover's output and ask an LLM (like ChatGPT) to translate the machine proof into concise, human-readable mathematical reasoning. What do you think about the output of the LLM?

References

Ertel, Wolfgang (2025). *Introduction to Artificial Intelligence*. 3rd. Springer. DOI: 10.1007/978-3-658-43102-0.

Godel, Kurt (1931). “Diskussion zur Grundlegung der Mathematik: Erkenntnis 2.” In: *Monatshefte fur Mathematik Und Physik* 32, pp. 147–148.