VNU-HUS MAT1206E/3508: Introduction to AI

Limitations of Logic

Hoàng Anh Đức

Bộ môn Tin học, Khoa Toán-Cơ-Tin học
Đại học KHTN, ĐHQG Hà Nội
hoanganhduc@hus.edu.vn

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The Search Space Problem

- In the search for a proof, depending on the calculus, potentially *there are (infinitely) many ways to apply inference rules at each step*
- This is the main reason for the *explosive growth of the search space*

Because of the search space problem, *automated provers* today can *only prove relatively simple theorems in special domains with few axioms*

The Search Space Problem

	Automated Provers	Human Experts
Number of inferences performed per second	20000	1
Solve difficult problems	slow	fast

Reasons

- Humans use *intuitive calculi* that work on a higher level
  - This intuitive calculi often *carry out many of the simple inferences of an automated prover in one step*
- Humans work with *lemmas*
  - We already know the lemmas are true and *do not need to re-prove* them each time
- Humans use *intuitive meta knowledge* (informal!)
  - Intuition is an important advantage of humans. Without it, we could not solve any difficult problems
- Humans use *heuristics*
  - In many cases, heuristics can greatly simplify or shorten the way to the goal

The Search Space Problem

Problems

- Often humans are *unable to formulate intuitive meta-knowledge verbally*!
- Humans *learn heuristics by experience*

Solution 1

Try to “copy nature”

- Learn heuristics by the application of machine learning techniques

Solution 2

Design interactive systems that operate under the control of the user

- For example, computer algebra programs such as Mathematica, Maple, or Maxima can automatically carry out difficult symbolic mathematical manipulations
- The search for the proof is left fully to the human

The Search Space Problem

Example 1 (Resolution + Machine Learning Techniques)

A *resolution prover* has, during the search for a proof, *hundreds or more possibilities for resolution steps at each step*, but *only a few lead to the goal*. It would be *ideal* if the prover could *ask an oracle which two clauses it should use in the next step to quickly find the proof*

(1) A *proof-directing module evaluates* the different alternatives for the next step heuristically and *chooses* the alternative with the *best rating*

- Rating the available clauses by *a function that calculates a value based on the number of literals, the number of positive literals, the complexity of the terms, etc.* for every pair of resolvable clauses

(2) How to *compute* such a *function*?

- Use machine learning algorithms to learn from successful proofs
- Successful resolution steps are stored as positive
- Unsuccessful resolution steps are stored as negative
- Machine learning system generates a program for the evaluation of clauses

Decidability and Incompleteness

- There are *correct and complete calculi and theorem provers*
- Any *theorem (i.e., a true statement)* can be *proved in a finite amount of time*
- What if *the statement is not true*?
  - There is *no process* that can *prove or refute any formula from PL1 in finite time*

Theorem 1

The set of valid formulas in first-order predicate logic is semidecidable

- This theorem implies that there are programs (theorem provers) which, given a true (valid) formula as input, determine its truth in finite time
- If the formula is not valid, however, it may happen that the prover never halts

Decidability and Incompleteness

Note

- *Propositional logic decidable* because *the truth table method provides all models of a formula in finite time*
- Evidently predicate logic with quantifiers and nested function symbols is a language somewhat too powerful to be decidable

Exercise 1 ([Ertel 2025], Exercise 4.1, p. 73)

(a) With the following (false) argument, one could claim that PL1 is decidable: *"We take a complete proof calculus for PL1. With it we can find a proof for any true formula in finite time. For every other formula ϕ I proceed as follows: I apply the calculus to $\neg\phi$ and show that $\neg\phi$ is true. Thus ϕ is false. Thus I can prove or refute every formula in PL1."* Find the mistake in the argument and change it so it becomes correct (**Hint:** There are three types of formulas: satisfiable, valid, and unsatisfiable. Can the above process be applied for all of these types?)

(b) Construct a decision process for the set of true and unsatisfiable formulas in PL1

Decidability and Incompleteness

Higher-order Logic

- A *first-order logic* can only *quantify over variables*
- A *second-order logic* can also *quantify over formulas of the first order*
- A *third-order logic* can *quantify over formulas of the second order*
- and so on

Example 2 (Second-order Logic Formula)

- If a predicate $p(n)$ holds for n, then $p(n+1)$ also holds
- $\forall p\, p(n) \Rightarrow p(n+1)$

Decidability and Incompleteness

Theorem 2 (Gödel incompleteness theorem)

Every axiom system for the natural numbers with addition and multiplication (arithmetic) is incomplete. That is, there are true statements in arithmetic that are not provable.

Proof Idea.

- Gödel's proof works with what is called *Gödelization*.
- Every arithmetic formula is encoded as a number (Gödel number).
- Gödelization is now used to formulate the proposition $F =$ *"I am not provable."* in the language of arithmetic.
- F *is true and not provable*.
  - Assume F is false. Then we can prove F and therefore show that F is not provable. This is a contradiction.

□

Decidability and Incompleteness

Note

The deeper background of Theorem 2 is that mathematical theories (axiom systems) and, more generally, *languages become incomplete if the language becomes too powerful* (e.g., PL1).

Example 3 ("Too powerful language")

- Set theory is so powerful that one can formulate paradoxes (= statements that contradict themselves) with it.
- For example, a paradox in set theory: "The set of all the barbers who all shave those who do not shave themselves"

Exercise 2 ([Ertel 2025], Exercise 4.2, p. 73)

(a) Given the statement "There is a barber who shaves every person who does not shave himself." Consider whether this barber shaves himself.

(b) Let $M = \{x \mid x \notin x\}$. Describe this set and consider whether M contains itself.

Dilemma: Languages which are powerful enough to describe mathematics and interesting applications also contain contradictions and incompletenesses

Example: The Flying Penguin

Goal

We present an example to demonstrate a fundamental problem of logic and possible solution approaches.

1. Tweety is a penguin
2. Penguins are birds
3. Birds can fly

Formalized in PL1, the *knowledge base* KB is:

$$\mathit{penguin}(\mathit{tweety})$$
$$\mathit{penguin}(x) \Rightarrow \mathit{bird}(x)$$
$$\mathit{bird}(x) \Rightarrow \mathit{fly}(x)$$

Exercise 3

(a) Show that $KB \vdash \mathit{fly}(\mathit{twetty})$ (for example, with resolution)

(b) Show that if we add $\mathit{penguin}(x) \Rightarrow \neg\mathit{fly}(x)$ to the knowledge base KB, then $KB \vdash \neg\mathit{fly}(\mathit{twetty})$

Example: The Flying Penguin

- From Exercise 3, both *fly*(*twetty*) and ¬*fly*(*twetty*) can be derived ⇒ *The knowledge base is inconsistent*
- Although we explicitly state that penguins cannot fly, the opposite can still be derived. This is because of *the monotony of PL1*.

Monotonic Logic

A logic is called *monotonic* if, for an arbitrary knowledge base KB and an arbitrary formula ϕ, the set of formulas derivable from KB is a subset of the formulas derivable from $KB \cup \phi$.

Recap

- Evidently the formalization of the flight attributes of penguins is insufficient
- To prevent the formula *fly*(*twetty*) from being derived, our first attempt is to add new formulas to the KB. This does not work

Example: The Flying Penguin

- We continue by replacing the obviously false statement "(all) birds can fly" in KB with the more exact statement "(all) birds except penguins can fly" and obtain as KB_2 the following clauses:

$$\begin{aligned} \mathit{penguin}(\mathit{tweety}) \\ \mathit{penguin}(x) &\Rightarrow \mathit{bird}(x) \\ \mathit{bird}(x) \wedge \neg \mathit{penguin}(x) &\Rightarrow \mathit{fly}(x) \\ \mathit{penguin}(x) &\Rightarrow \neg \mathit{fly}(x) \end{aligned}$$

- **Problem solved!** We *can now derive* $\neg \mathit{fly}(\mathit{tweety})$ *but not* $\mathit{fly}(\mathit{tweety})$, because to derive $\mathit{fly}(\mathit{tweety})$ we would need $\neg \mathit{penguin}(x)$, which is not derivable

Example: The Flying Penguin

- A problem arises when we want to add a new bird, say the raven Abraxas (from the German book "The Little Witch"), and obtain KB_3

$$raven(abraxas)$$
$$raven(x) \Rightarrow bird(x)$$
$$penguin(tweety)$$
$$penguin(x) \Rightarrow bird(x)$$
$$bird(x) \wedge \neg penguin(x) \Rightarrow fly(x)$$
$$penguin(x) \Rightarrow \neg fly(x)$$

- At the moment, we cannot say anything about the flight attributes of Abraxas because we forgot to formulate that ravens are not penguins. Thus we extend KB_3 to KB_4

$$raven(abraxas)$$
$$raven(x) \Rightarrow bird(x)$$
$$raven(x) \Rightarrow \neg penguin(x)$$
$$penguin(tweety)$$
$$penguin(x) \Rightarrow bird(x)$$
$$bird(x) \wedge \neg penguin(x) \Rightarrow fly(x)$$
$$penguin(x) \Rightarrow \neg fly(x)$$

Example: The Flying Penguin

- The fact that ravens are not penguins, which is self-evident to humans, must be explicitly added here
- For the construction of a knowledge base with all 9800 or so types of birds worldwide, it must therefore be specified for every type of bird (except for penguins) that it is not a member of penguins
- *In general, for every object in the knowledge base, in addition to its attributes, all of the attributes it does not have must be listed.*

Exercise 4 ([Ertel 2025], Exercise 4.3, p. 73)

Use an automated theorem prover (for example E [Schulz 2002]) and apply it to all five different axiomatizations of the Tweety example mentioned above. Validate the example's statements.

Example: The Flying Penguin

Another problem caused by the monotony is the so-called *frame problem*. This happens in complex planning problems in which the world can change

Example 4 (An example of the frame problem)

- A blue house is painted red, then afterwards it is red
- However, with the knowledge base

$$\begin{aligned} \mathit{color}(\mathit{house}, \mathit{blue}) \\ \mathit{paint}(\mathit{house}, \mathit{red}) \\ \mathit{paint}(x, y) \Rightarrow \mathit{color}(x, y) \end{aligned}$$

one can derive $\mathit{color}(\mathit{house}, \mathit{red})$

- Additionally, $\mathit{color}(\mathit{house}, \mathit{blue})$ *is already in the knowledge base, which leads to the conclusion that, after painting, the house is both blue and red*

Example: The Flying Penguin

- To solve this problem, *non-monotonic logics* have been developed.
  - Knowledge (formulas) can be removed from the knowledge base.
- Despite great effort, these logics have at present, due to semantic and practical problems, not succeeded.
- Another interesting approach for modeling problems such as the Tweety example is *probability theory*.
  - The statement “all birds can fly” is false.
  - A statement something like “almost all birds can fly” is correct.
  - This statement becomes more exact if we give a probability for “birds can fly”.

Modelling Uncertainty

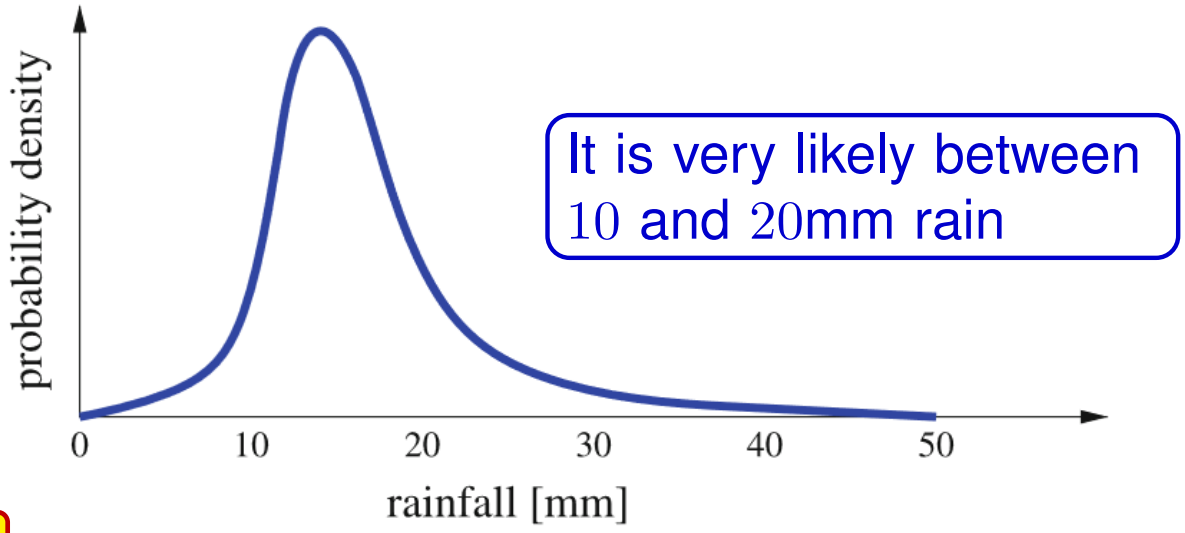
- *Two-valued logic* can and should only *model circumstances in which there is true, false, and no other truth values*.
- For many tasks in everyday reasoning, *two-valued logic is therefore not expressive enough*.
  - For example, the rule $bird(x) \Rightarrow fly(x)$ is *true for almost all birds, but for some it is false*.
- As we already mentioned, to formulate uncertainty, we can use *probability theory*.
  - For example, we give a probability for *"birds can fly"*: $P(bird(x) \Rightarrow fly(x)) = 0.99$ *(i.e., "*99% *of all birds can fly")*
  - Later, we will see that here it is better to work with *conditional probabilities* such as $P(fly|bird) = 0.99$. With the help of *Bayesian networks*, complex applications with many variables can also be modelled.
  - *Fuzzy logic* is required for *"The weather is nice"*. Here it makes no sense to speak in terms of true and false.
  - The variable *weather_is_nice* is *continuous with values in* $[0, 1]$. *weather_is_nice* $= 0.7$ then means *"The weather is fairly nice"*.

Modelling Uncertainty

- Probability theory also offers the possibility of *making statements about the probability of continuous variables*.

"There is a high probability that there will be some rain"

$$P(\textit{rainfall} = X) = Y$$



Note

This very *general and even visualizable representation* of both types of uncertainty we have discussed, together with *inductive statistics* and the theory of *Bayesian networks*, makes it possible, in principle, *to answer arbitrary probabilistic queries*.

Modelling Uncertainty

Comparison of different formalisms for the modelling of uncertain knowledge

Formalism	**Number of truth values**	**Probabilities expressible**
Propositional logic	2	—
Fuzzy logic	∞	—
Discrete probabilistic logic	n	yes
Continuous probabilistic logic	∞	*yes*

References

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