

VNU-HUS MAT1206E/3508: Introduction to AI

Limitations of Logic

In-class Discussion

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The Search Space Problem

Decidability and Incompleteness

Example: The Flying Penguin

Modelling Uncertainty

- In the search for a proof, depending on the calculus, potentially *there are (infinitely) many ways to apply inference rules at each step*
- This is the main reason for the *explosive growth of the search space*



Because of the search space problem, *automated provers* today can *only prove relatively simple theorems in special domains with few axioms*

The Search Space Problem



Human experts can prove theorems that are *out of reach* for *automated provers*

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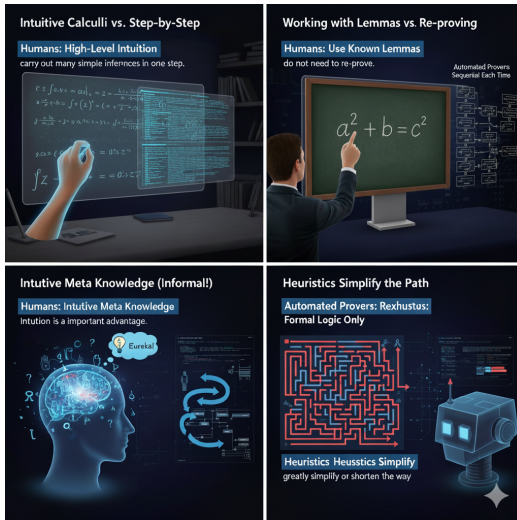
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Modelling Uncertainty

Human experts vs.
Automated Provers.
Image generated by
Gemini AI



The Search Space Problem



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Modelling Uncertainty

Problems?

- Humans' *intuitive meta-knowledge* is difficult to formalize and integrate into automated provers.
- Humans acquire *heuristics through experience*, and these heuristics are hard to transfer to automated provers.

Possible Solutions?

- *Machine learning* techniques can be used to learn heuristics from large datasets of proofs.
- *Hybrid approaches* that combine automated reasoning with human expertise can be explored.

Decidability and Incompleteness



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Example: The Flying
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Modelling Uncertainty

- **Decidability** refers to whether there exists an algorithm that can determine the truth or falsity of any statement in a given logical system.
- **Incompleteness** refers to the fact that in any sufficiently powerful logical system, there are statements that are true but cannot be proven within the system.

Examples?

- The **Halting Problem** is a classic example of an undecidable problem.
- In first-order logic, if a given formula is valid, there is a proof of it (Gödel's Completeness Theorem). However, if the formula is not valid, it may happen that the prover never halts (undecidability). On the other hand, propositional logic is decidable. **[Why?]**
- Gödel's **Incompleteness Theorems** show that in any consistent formal system that is capable of expressing basic arithmetic, there are true statements that cannot be proven within the system.

Decidability and Incompleteness



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Note

The deeper background of Gödel's Incompleteness Theorem is that mathematical theories (axiom systems) and, more generally, *languages become incomplete if the language becomes too powerful* (e.g., PL1).

Example 1 ("Too powerful language")

- *Set theory* is so powerful that one can formulate paradoxes (= statements that contradict themselves) with it.
- For example, a paradox in set theory: "The set of all the barbers who all shave those who do not shave themselves"

Example: The Flying Penguin



Knowledge Base (KB)

- (1) Tweety is a penguin
- (2) Penguins are birds
- (3) Birds can fly

- (1) $penguin(Tweety)$
- (2) $\forall x (penguin(x) \Rightarrow bird(x))$
- (3) $\forall x (bird(x) \Rightarrow fly(x))$



Query (Q)

Can *Tweety fly*? ($Q = fly(Tweety)$)

Note that $KB \vdash Q$, i.e., Q can be proven from KB **[How?]**

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However, in reality, *penguins cannot fly*. That is, we should prevent the derivation of Q from KB

First Attempt

To mimics the reality, we try to add $\forall x (penguin(x) \Rightarrow \neg fly(x))$ to KB .

- After adding this new information, we have $KB \vdash \neg Q$, i.e., $\neg Q$ can be proven from KB **[How?]**
- But Q can still be derived from KB **[How?]**
- Both Q and $\neg Q$ can be derived from $KB \Rightarrow$ *contradiction*
 $\Rightarrow KB$ is *inconsistent* \Rightarrow *everything can be proven from*
KB (principle of explosion)
 - **Reason:** PL1 is *monotonic*, i.e., if $KB \vdash Q$, then
 $(KB \cup \{R\}) \vdash Q$ for any formula R .

\Rightarrow Does not work!

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Second Attempt

- Add “Penguins cannot fly” to the original KB
- Replace the non-realistic rule “Birds can fly” by “Birds except penguins can fly”.

Knowledge Base (KB_2)

- | | |
|-----------------------------------|---|
| (1) Tweety is a penguin | (1) $penguin(Tweety)$ |
| (2) Penguins are birds | (2) $\forall x (penguin(x) \Rightarrow bird(x))$ |
| (3) Birds except penguins can fly | (3) $\forall x (bird(x) \wedge \neg penguin(x) \Rightarrow fly(x))$ |
| (4) Penguins cannot fly | (4) $\forall x (penguin(x) \Rightarrow \neg fly(x))$ |

\Rightarrow Problem solved! Now, $\neg Q$ can be derived from KB_2 , but Q cannot

Example: The Flying Penguin



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But ...

- Whenever we want to add a new bird, we have to specify whether it can fly or not
- This means we have to explicitly specify a rule saying that the newly added bird is not a penguin (because of the rule “Birds except penguins can fly”); otherwise, we have no conclusion about whether the new bird can fly or not
- For the construction of a knowledge base with all 9800 or so types of birds worldwide, this becomes a significant challenge.

In general, for every object in the knowledge base, in addition to its attributes, all of the attributes it does not have must be listed.

Example: The Flying Penguin



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Another problem caused by the monotony is the so-called *frame problem*. This happens in complex planning problems in which the world can change

Example 2 (An example of the frame problem)

- A blue house is painted red, then afterwards it is red
- However, with the knowledge base

color(house, blue)

paint(house, red)

paint(x, y) \Rightarrow color(x, y)

one can derive *color(house, red)*

- Additionally, *color(house, blue)* is already in the knowledge base, which leads to the conclusion that, after painting, the house is both blue and red

Example: The Flying Penguin



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Modelling Uncertainty

Some Resolving Ideas

- *Non-monotonic reasoning* allows us to withdraw conclusions in the light of new information (i.e., old knowledge can be removed).
- *Probability theory* can be used to handle uncertainty and make decisions based on incomplete information.

Modelling Uncertainty



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- *Two-valued logic* can and should only *model circumstances in which there is true, false, and no other truth values*.
- For many tasks in everyday reasoning, *two-valued logic is therefore not expressive enough*.
 - For example, the rule $\text{bird}(x) \Rightarrow \text{fly}(x)$ is *true for almost all birds, but for some it is false*.
- As we already mentioned, to formulate uncertainty, we can use *probability theory*.
 - For example, we give a probability for "*birds can fly*":
 $P(\text{bird}(x) \Rightarrow \text{fly}(x)) = 0.99$ (i.e., "99% of all birds can fly")
 - Later, we will see that here it is better to work with *conditional probabilities* such as $P(\text{fly}|\text{bird}) = 0.99$. With the help of *Bayesian networks*, complex applications with many variables can also be modelled.
 - *Fuzzy logic* is required for "*The weather is nice*". Here it makes no sense to speak in terms of true and false.
 - The variable *weather_is_nice* is *continuous with values in $[0, 1]$* . $\text{weather_is_nice} = 0.7$ then means "*The weather is fairly nice*".

Modelling Uncertainty



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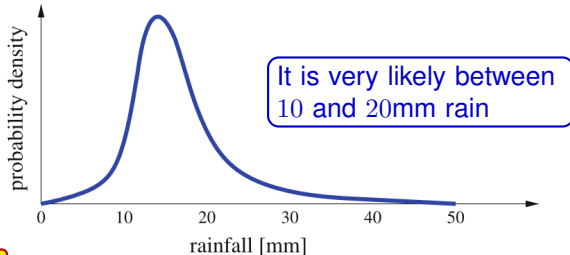
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- Probability theory also offers the possibility of *making statements about the probability of continuous variables*.

“There is a high probability that there will be some rain”

$$P(\text{rainfall} = X) = Y$$



Note

This very *general and even visualizable representation* of both types of uncertainty we have discussed, together with *inductive statistics* and the theory of *Bayesian networks*, makes it possible, in principle, *to answer arbitrary probabilistic queries*.

Modelling Uncertainty



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Comparison of different formalisms for the modelling of
uncertain knowledge

Formalism	Number of truth values	Probabilities expressible
Propositional logic	2	—
Fuzzy logic	∞	—
Discrete probabilistic logic	n	yes
<i>Continuous probabilistic logic</i>	∞	<i>yes</i>

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Modelling Uncertainty