The Complexity of Distance-r Dominating Set Reconfiguration

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Joint work with Duc A. Hoang

Model: Graph Reconfiguration

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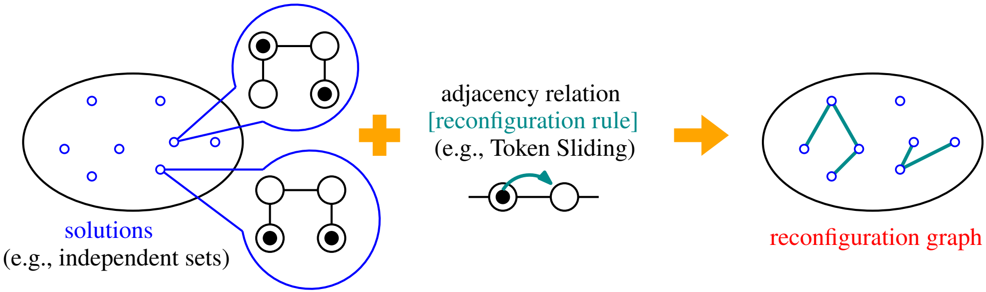

Figure: Reconfiguration

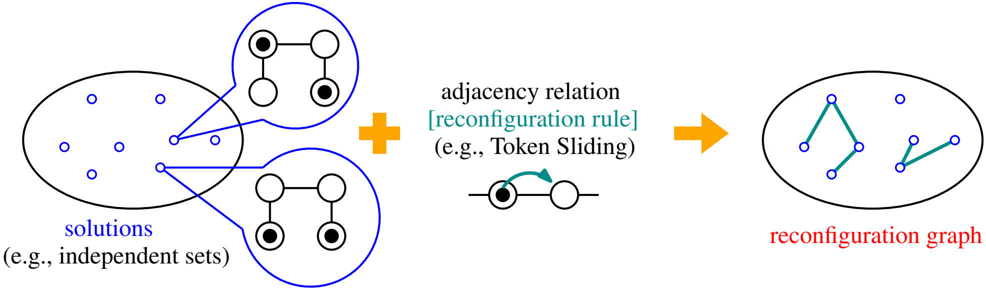

Figure: Reconfiguration

Main questions:

- REACHABILITY: Is there a path between two given solutions? Can we transform S into T via a sequence of feasible solutions.

  Such a sequence, if exists, is called a *reconfiguration sequence*.

- SHORTEST PATH: If REACHABILITY is yes, can we find a shortest path between S and T?

Reconfiguration rules: Tells us rules we need to follow to go from one feasible solution to another.

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A few well known Reconfiguration rules: Token Sliding (TS), Token Jumping (TJ) and Token Addition/ Removal (TAR).

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Reconfiguration rules of Token Sliding (TS), Token Jumping (TJ) and Token Addition/ Removal (TAR).

- **Token Sliding (TS):** one can move a token to one of its unoccupied neighbors as long as the resulting token-set forms a feasible solution.
- **Token Jumping (TJ):** one can move a token to any unoccupied vertex as long as the resulting token-set forms a feasible solution.
- **Token Addition/Removal ($TAR(k)$):** one can either add or remove a token as long as the resulting token-set forms a feasible solution of size at most some threshold $k \geq 0$.

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Distance-r Dominating Set Reconfiguration DrDSR

DrDSR: We solve Reachability of DrDS in the reconfiguration graph.

DrDSR

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Previous Results :

- Haddadan et al.[TCS2016] first studied the computational complexity of Dominating Set Reconfiguration (DSR) under TAR and showed that the problem is $PSPACE$-complete on planar graphs of maximum degree six, bounded bandwidth graphs, split graphs and bipartite graphs.

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- Haddadan et al.[TCS2016] designed polynomial-time algorithms for solving DSR under TAR on cographs, forests, and interval graphs. Bonamy et al.[DAM2021] showed the above mentioned results also hold under TJ.

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- Further results on DSR was also shown in Bousquet and Joffard[FCT2021] and Křišt'an and Svoboda[FCT2023].

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Previous Results (Parameterized variant):

There are two natural parameterizations: the number of tokens k and the length of a reconfiguration sequence ℓ.

- Mouawad et al.[ALG2017] showed that DSR under TAR on general graphs is $\texttt{W}[1]$-hard parameterized by k and $\texttt{W}[2]$-hard parameterized by $k + \ell$.

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- When parameterized by k on graphs excluding $K_{d,d}$ as a subgraph, Lokshtanov et al.[JCSS2018] designed an FPT algorithm for solving the problem.
- When parameterized by ℓ alone, it was mentioned in Bousquet and Joffard[FCT2021] that the problem is fixed-parameter tractable on any class where first-order model-checking is fixed-parameter tractable.

DrDSR

Our Results:

We prove hardness results of DrDSR in different graph classes for $r \geq 2$ under reconfiguration rule of Token Sliding (TS) and Token Jumping (TJ).

Distance-r Dominating Set

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In other graph classes we show DrDSR, $r \geq 2$ is solvable in polynomial time.

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- DrDSR can be solved in polynomial time on split graphs under both TS and TJ.
- DrDSR can be solved in polynomial time on trees under TJ.
- DrDSR can be solved in polynomial time on interval graphs under TJ and co-graphs under both TS and TJ.

DrDSR

Theorem

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Proof Sketch:

Let (G, C_s, C_t) be an instance of $\mathrm{M\text{-}VCR}$ where C_s, C_t are two minimum VCs of a graph G.

We will construct an instance (G', D_s, D_t) of $\mathrm{D}r\mathrm{DSR}$ where D_s and D_t are two DrDSs of a bipartite graph G'.

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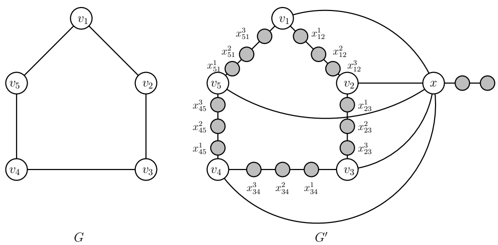

Figure: Lower Bound on Bipartite Graphs

Suppose that $V(G) = \{v_1, \ldots, v_n\}$. We construct G' from G as follows.

a. Replace each edge $v_i v_j$ by a path $P_{ij} = x_{ij}^0 x_{ij}^1 \ldots x_{ij}^{2r}$ of length $2r$ $(1 \leq i, j \leq n)$ with $x_{ij}^0 = v_i$ and $x_{ij}^{2r} = v_j$.

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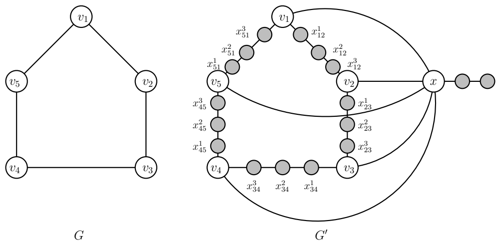

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Suppose that $V(G) = \{v_1, \ldots, v_n\}$. We construct G' from G as follows.

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b. Add a new vertex x and join it to every vertex in $V(G)$.

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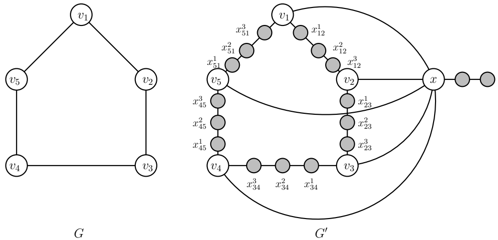

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- b Add a new vertex x and join it to every vertex in $V(G)$.
- c Attach a new path P_x of length r to x.

We define $D_s = C_s + x$ and $D_t = C_t + x$.

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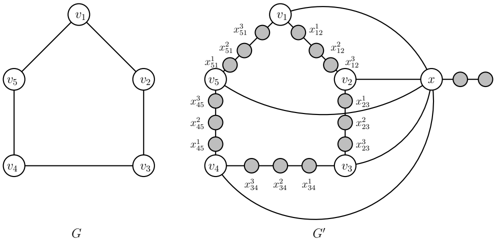

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G' is a bipartite graph.

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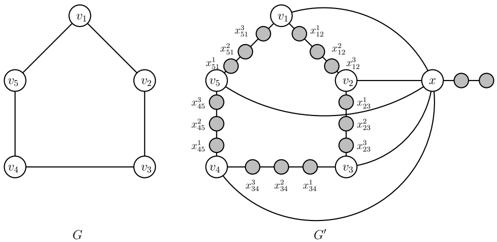

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Any set of the form $C + x$, where C is a minimum VC of G, is a minimum DrDS of G'.

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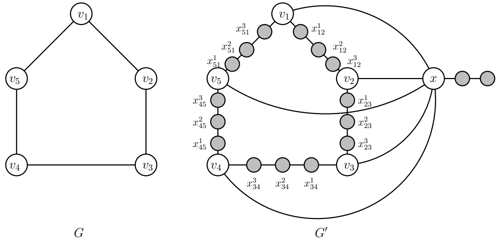

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Any set of the form $C + x$, where C is a minimum VC of G, is a minimum Dr DS of G'.

Lemma

(G, C_s, C_t) is a yes-instance if and only if (G', D_s, D_t) is a yes-instance.

Theorem

We show that $\mathrm{D}r\mathrm{DSR}$ *under* TJ *on trees runs in polynomial time for any* $r \geq 2$*.*

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Proof Sketch:

Let the tree be T.

- Based on the minimum DrDS, $D^{\star}$ obtained from the implementation of Abu-Affash, Carmi, and Krasin[DAM2022], we construct a partition $\mathbb{P}(T)$ of T consisting of vertex-disjoint subtrees.

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- To find $\mathbb{P}(T)$: Let T be rooted at u. In each iteration, it finds a subtree T_v of height exactly r, adds v to $D^{\star}$, and removes all the leaves of T_u that are in $N^r_{T_u}[v]$.

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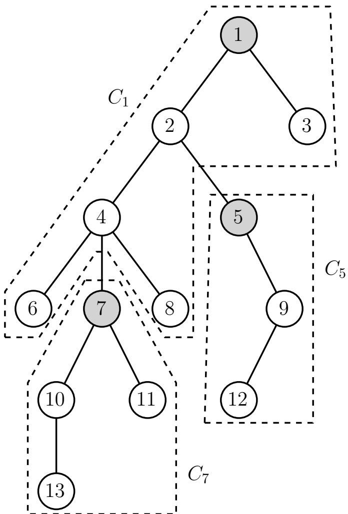

Figure: A tree T_u rooted at $u = 1$. For $r = 2$, Algorithm returns $D^\star = \{7, 5, 1\}$. A partition $\mathbb{P}(T_u) = \{C_7, C_5, C_1\}$ of T_u is also constructed.

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Let D *be an arbitrary D*r*DS of* T_u. *Let* D' *be any D*r*DS of* T_u *that contains* $D^\star$, *i.e.,* $D^\star \subseteq D'$. *Then, in* $O(n)$ *time, one can construct a* TJ*-sequence* S *in* T_u *between* D *and* D'.

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- The key idea is one can transform both D_s and D_t into some DrDS D that contains $D^\star$. For instance, to transform D_s into D, for each subtree $C_x \in \mathbb{P}(T)$ for $x \in D^\star$, we move any token in $D_s \cap V(C_x)$ to x.

DrDSR

Open Questions:

- What is the complexity of DrDSR, $r \geq 2$ under TS on trees?
- What is the complexity of DrDSR, $r \geq 2$ under TS on interval graphs?

Thank You!