AFSA B01 Group: SSSS 2022.05

On Reconfiguration Graph of Independent Sets under Token Sliding

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(A research starting from SSSW 2021.04 @ Kyoto)

Outline

TS-Reconfiguration Graphs

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- s-Partite
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Open Questions

- Can Forests Be TS$_k$-Graphs?

Given a graph $G = (V, E)$ and a positive integer k.

	G
Vertex	$V(G)$
Edge	$E(G)$

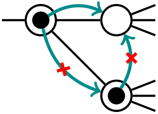
- Each vertex of G contains at most *one unlabeled* token.

TS-Reconfiguration Graphs
Introduction

Given a graph $G = (V, E)$ and a positive integer k.

	G
Vertex	$V(G)$
Edge	$E(G)$

- Each vertex of G contains at most *one unlabeled* token.
- *Token Sliding* involves moving a token from one vertex to one of its *unoccupied adjacent* vertices.



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TS-Reconfiguration Graphs

Introduction

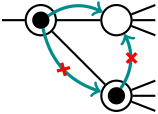
Given a graph $G = (V, E)$ and a positive integer k.

	G	*Token Graph* $F_k(G)$
Vertex	$V(G)$	k-Vertex Subsets of $V(G)$
Edge	$E(G)$	Token Sliding

Well-known variants of $F_k(G)$ involve 15-PUZZLE, PEBBLE MOTION, CHIP-FIRING GAME, ROBOT MOTION, and so on

$F_2(G)$ [Alavi, Behzad, Erdős, and Lick 1991]

- Each vertex of G contains at most *one unlabeled* token.
- *Token Sliding* involves moving a token from one vertex to one of its *unoccupied adjacent* vertices.



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Given a graph $G = (V, E)$ and a positive integer k.

	G	*Token Graph* $F_k(G)$	TS_k*-(Reconfiguration) Graph* $\mathsf{TS}_k(G)$
Vertex	$V(G)$	k-Vertex Subsets of $V(G)$	*Independent* k-Vertex Subsets of $V(G)$
Edge	$E(G)$	Token Sliding	Token Sliding

$F_2(G)$ [Alavi, Behzad, Erdős, and Lick 1991] [This Talk]

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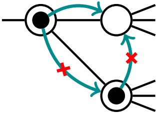

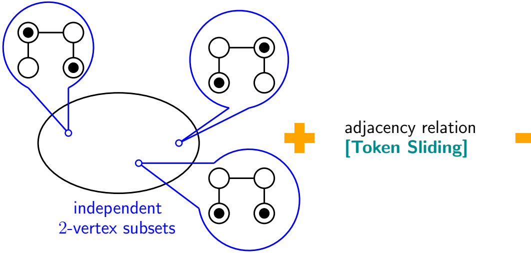
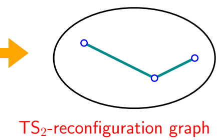

Figure: $\mathsf{TS}_2(P_4) \subseteq F_2(P_4)$

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In this talk, we focus on two graphs:

- $\mathsf{TS}_k(G)$ whose nodes are independent sets of *size-k*, and
- $\mathsf{TS}(G)$ whose nodes are independent sets of *arbitrary size*.

Note

- $\mathsf{TS}_k(G)$ is an induced subgraph of $\mathsf{TS}(G)$.
- $\mathsf{TS}_k(G)$ may *not* be a component of $\mathsf{TS}(G)$.
- $k = 1$ is not interesting, since $G \simeq \mathsf{TS}_1(G)$ for any graph G. Therefore *we always consider $k \geq 2$*.

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Reconfiguration Setting

- Moving from one *state/configuration* to another
  - The whole state space is *not* given, but you can, in polynomial-time, check if a state is "valid".
- ... using a pre-defined *(reconfiguration) rule*
  - You can, in polynomial-time, check if one "valid" state can be transformed to another by using the rule *exactly once*.

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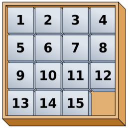
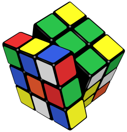
A More General Framework

Reconfiguration Setting

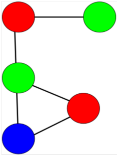
- Moving from one *state/configuration* to another
- ... using a pre-defined *(reconfiguration) rule*

- Games & Puzzles.

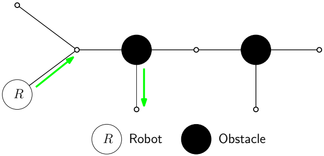




- Frequency Re-Assignment.



- Robot Motion.



TS-Reconfiguration Graphs
Motivation

- "Reconfiguration of Independent Sets" is one of the most well-studied topics in the Reconfiguration framework.
- Most of the previous research on $\mathsf{TS}_k(G)$ have been focused on *designing efficient algorithms* and *showing computational hardness* of several reconfiguration questions [Nishimura 2018].
  - REACHABILITY/SHORTEST TRANSFORMATION: Is there a (shortest) path between two given nodes of $\mathsf{TS}_k(G)$?
  - CONNECTIVITY: Is $\mathsf{TS}_k(G)$ connected?
  - and so on.

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  - REACHABILITY/SHORTEST TRANSFORMATION: Is there a (shortest) path between two given nodes of $\mathsf{TS}_k(G)$?
  - CONNECTIVITY: Is $\mathsf{TS}_k(G)$ connected?
  - and so on.

Our Goal

We look at $\mathsf{TS}_k(G)$ from *a purely graph-theoretic viewpoint*.

- What are the necessary and sufficient conditions for $\mathsf{TS}_k(G)$ to be in some graph class $\mathcal{G}$?
- If G satisfies some property $\mathcal{P}$, does $\mathsf{TS}(G)/\mathsf{TS}_k(G)$ also satisfy $\mathcal{P}$, and vice versa?

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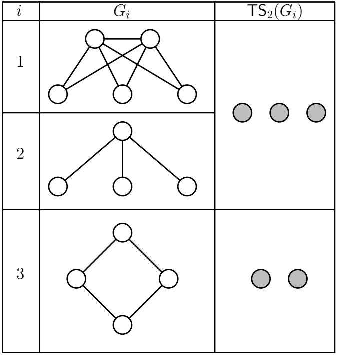
TS-Reconfiguration Graphs
Note

- If H and G are isomorphic, then so are $\mathsf{TS}_k(H)$ and $\mathsf{TS}_k(G)$. *The reverse does not hold*.
- If H is an induced subgraph of G, then $\mathsf{TS}_k(H)$ is an induced subgraph of $\mathsf{TS}_k(G)$. *The reverse does not hold*.



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Main Question

Is G a TS_k-graph?

More precisely, does there exist a graph H such that $G \simeq_f \mathsf{TS}_k(H)$, where $f: V(G) \to V(\mathsf{TS}_k(H))$ is bijective and $uv \in E(G)$ if and only if $f(u)f(v) \in E(\mathsf{TS}_k(H))$?

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Our Results

- There is a relationship between the *line graph of G* and the *TS_2-graph of its complement $\overline{G}$*.
- All *complete graphs*, *paths*, and *cycles* are TS_k-graphs.
- Characterize which *complete bipartite graphs* and *connected split graphs* are TS_k-graphs.

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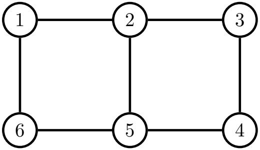
Realizing TS$_k$-Reconfiguration Graphs

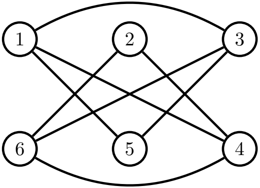
Line graph $L(G)$ and TS$_2$-graph TS$_2(\overline{G})$

Lemma 1

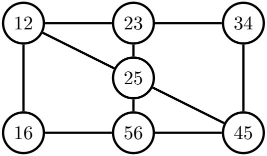
- TS$_2(\overline{G})$ *is a (spanning) subgraph of* $L(G)$.
- TS$_2(\overline{G}) \simeq L(G)$ *if and only if* G *is triangle-free.*



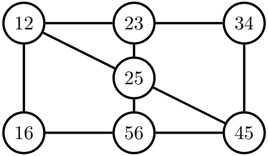
G



$\overline{G}$



$L(G)$



TS$_2(\overline{G})$

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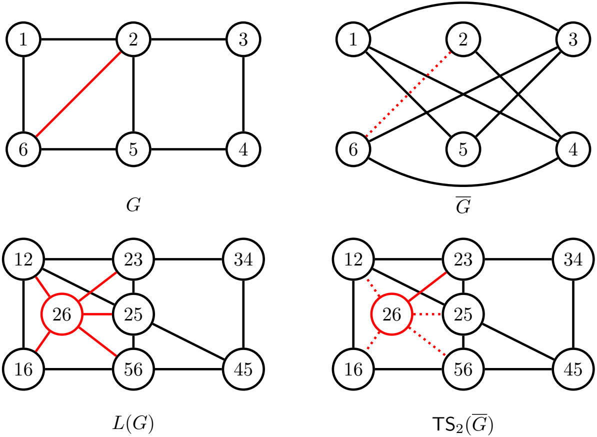
Realizing TS_k-Reconfiguration Graphs

Line graph $L(G)$ and TS_2-graph $\mathsf{TS}_2(\overline{G})$

Lemma 1

- $\mathsf{TS}_2(\overline{G})$ *is a (spanning) subgraph of* $L(G)$.
- $\mathsf{TS}_2(\overline{G}) \simeq L(G)$ *if and only if* G *is triangle-free.*



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Complete Graphs

Lemma 2

Given H and let $G = \mathsf{TS}_{\alpha(H)}(H)$. Then, for every $k \geq \alpha(H)$, G is a TS_k-graph.

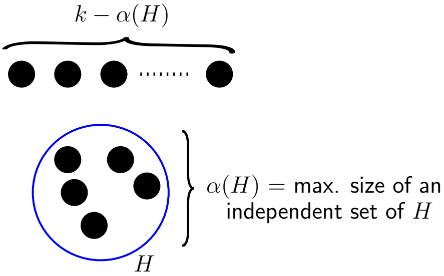

Figure: A graph H' s.t. $G = \mathsf{TS}_{\alpha(H)}(H) \simeq \mathsf{TS}_k(H')$ for $k \geq \alpha(H)$.

Corollary 3

K_n is a TS_k-graph for every $k \geq 1$.

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Proposition 4

P_n is a TS_k-graph.

- (suggested by Yuni IWAMASA.) For $k = 2$, use Lemma 1.

$$P_n = L(P_{n+1}) \simeq \mathsf{TS}_2(\overline{P_{n+1}}).$$

- For $k \geq 3$, note that $\alpha(\overline{P_{n+1}}) = 2$. Use Lemma 2.

Proposition 5

C_n is a TS_k-graph ($n \geq 4$).

- For $k = 2$, use Lemma 1.

$$C_n = L(C_n) \simeq \mathsf{TS}_2(\overline{C_n}).$$

- For $k \geq 3$, note that $\alpha(\overline{C_n}) = 2$. Use Lemma 2.

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Realizing TS$_k$-Reconfiguration Graphs
A Different Approach for Paths

Key Idea ($k = 2$)

There exists G such that $P_1 \simeq \mathsf{TS}_2(G)$. For $n \geq 2$, we *update G and label vertices of P_n by size-2 independent sets of G.*

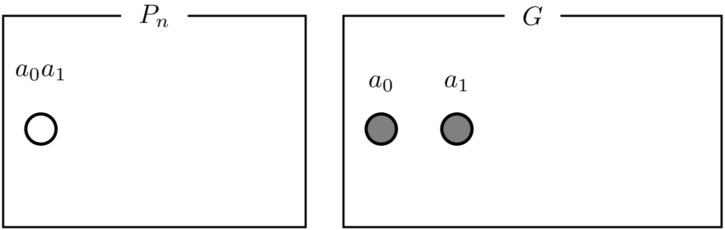

Figure: P_n is a TS$_2$-graph.

Note: $G \simeq \overline{P_{n+1}}$.

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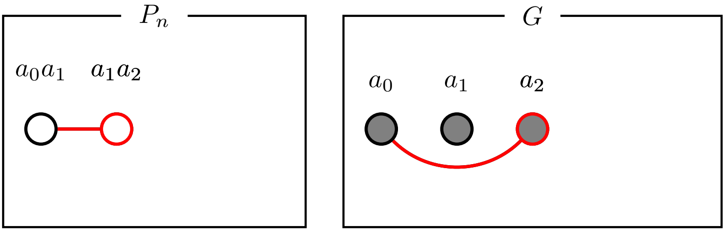

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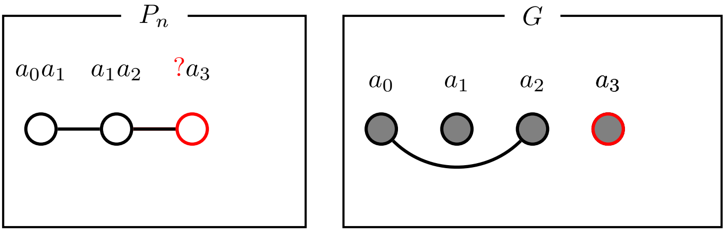

Figure: P_n is a TS_2-graph.

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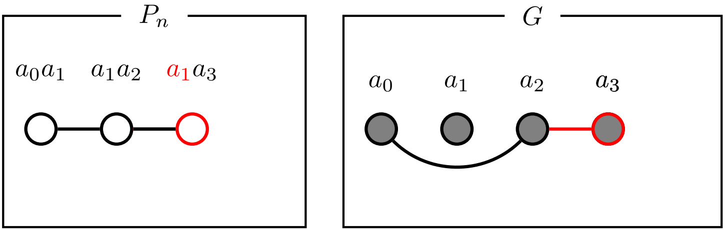

Figure: P_n is a TS_2-graph.

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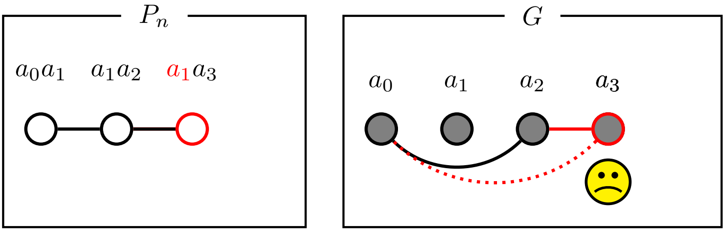

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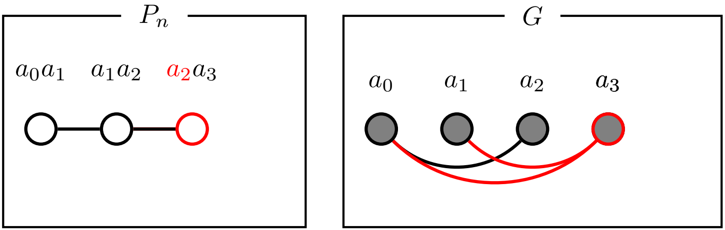

Figure: P_n is a TS_2-graph.

Note: $G \simeq \overline{P_{n+1}}$.

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A Different Approach for Paths

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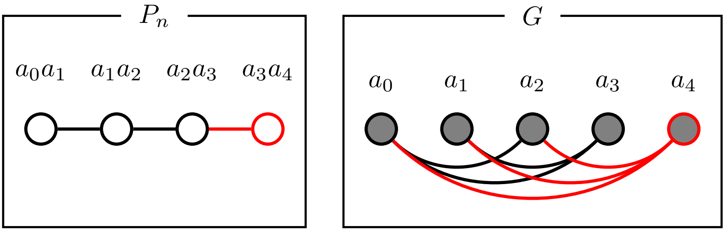

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Proposition 6

$K_{m,n}$ ($m \leq n$) is a TS_k-graph $\Leftrightarrow$ either $m = 1$ and $n \leq k$ or $m = n = 2$.

($\Leftarrow$) Since $K_{2,2} \simeq C_4$, the case $m = n = 2$ is trivial. If $m = 1$ and $n \leq k$, we construct G such that $K_{m,n} \simeq \mathsf{TS}_k(G)$.

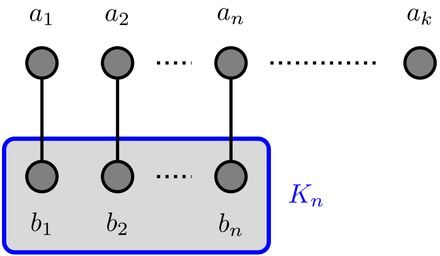

Figure: A graph G such that $K_{1,n} \simeq \mathsf{TS}_k(G)$ where $n \leq k$.

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$(\Rightarrow)$ Suppose to the contrary that there exists G such that $K_{m,n} \simeq \mathsf{TS}_k(G)$ where either $m = 1$ *and* $n \geq k + 1$ or $m \geq 2$ *and* $n > 2$.

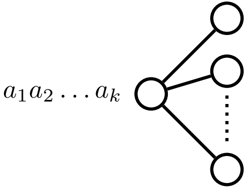

Figure: $K_{m,n}$ is not a TS_k-graph when $m = 1$ and $n \geq k + 1$.

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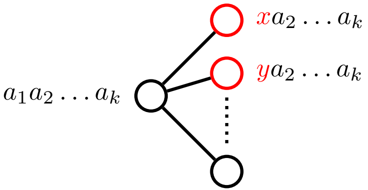

Pigeonhole principle

Figure: $K_{m,n}$ is not a TS_k-graph when $m = 1$ and $n \geq k + 1$.

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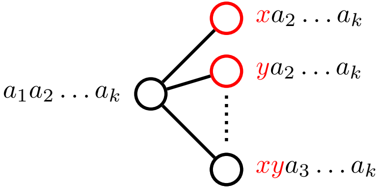

Pigeonhole principle

How about $x y a_3 \dots a_k$? ☹

Figure: $K_{m,n}$ is not a TS$_k$-graph when $m = 1$ and $n \geq k + 1$.

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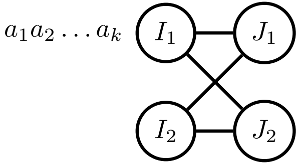

How can we label the rest?

Figure: $K_{m,n}$ is not a TS_k-graph when $m \geq 2$ and $n > 2$.

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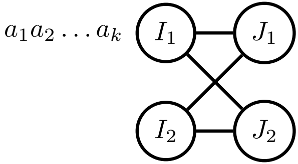

How can we label the rest?

At most two tokens can move from their original positions

Figure: $K_{m,n}$ is not a TS$_k$-graph when $m \geq 2$ and $n > 2$.

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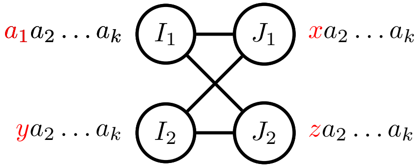

How can we label the rest?

If *one token* can move from its original position

Figure: $K_{m,n}$ is not a TS_k-graph when $m \geq 2$ and $n > 2$.

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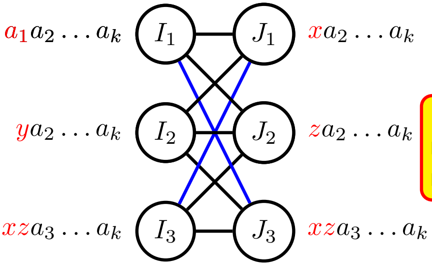

How about $xz a_3 \ldots a_k$? ☹

If *one token* can move from its original position

Figure: $K_{m,n}$ is not a TS$_k$-graph when $m \geq 2$ and $n > 2$.

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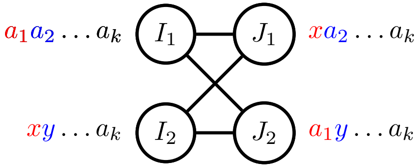

How can we label the rest?

If *two tokens* can move from their original position

Figure: $K_{m,n}$ is not a TS_k-graph when $m \geq 2$ and $n > 2$.

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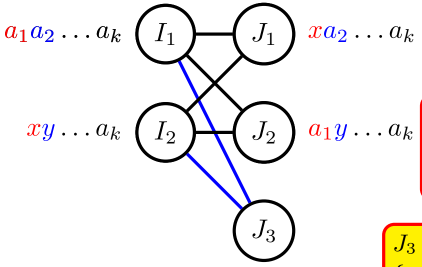

How can we label J_3? ☹

If *two tokens* can move from their original position

J_3 contains one member of $\{a_1, a_2\}$ and one of $\{x, y\}$.

Figure: $K_{m,n}$ is not a TS$_k$-graph when $m \geq 2$ and $n > 2$.

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Main Question

If G satisfies a graph property $\mathcal{P}$, does $\mathsf{TS}(G)$/$\mathsf{TS}_k(G)$ satisfy $\mathcal{P}$ too, and vice versa?

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Our Results

We answered the question for the following properties $\mathcal{P}$:

- s-Partite
- Planar
- Eulerian
- (Non-)acyclic
- The clique's size

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Our Results

<table>
<tr><th>$\mathcal{P}$</th><th>G</th><th>(a)</th><th>(b)</th><th>(c)</th><th>(d)</th></tr>
<tr><td>s-partite</td><td>general</td><td colspan="3">yes</td><td>no</td></tr>
<tr><td rowspan="4">planar</td><td>P_n</td><td colspan="2">yes, iff $n \leq 8$</td><td colspan="2">yes, iff $k = 2, n \geq 3$ or $k \geq 3, n \leq 8$</td></tr>
<tr><td>tree</td><td colspan="4">yes, iff $n \leq 7$</td></tr>
<tr><td>C_n</td><td colspan="4" rowspan="2">yes, iff $n \leq 6$</td></tr>
<tr><td>connected</td></tr>
<tr><td rowspan="2">Eulerian</td><td>C_n</td><td>no</td><td>yes</td><td colspan="2">yes</td></tr>
<tr><td>general</td><td>no</td><td>yes</td><td>no</td><td>no</td></tr>
<tr><td>acyclic</td><td>P_n</td><td colspan="4">yes, iff $n \leq 4$</td></tr>
<tr><td>non-acyclic</td><td>C_n</td><td>no</td><td colspan="3">yes, iff $1 \leq k < n/2$</td></tr>
<tr><td>having K_s</td><td>general</td><td colspan="2">yes</td><td>no</td><td>yes</td></tr>
</table>

Table: Some properties of (reconfiguration) graphs. Here $n = |V(G)|$. There are four cases: (a) $\mathcal{P}(G) \Rightarrow \mathcal{P}(\mathsf{TS}(G))$, (b) $\mathcal{P}(\mathsf{TS}(G)) \Rightarrow \mathcal{P}(G)$, (c) $\mathcal{P}(G) \Rightarrow \mathcal{P}(\mathsf{TS}_k(G))$, and (d) $\mathcal{P}(\mathsf{TS}_k(G)) \Rightarrow \mathcal{P}(G)$. ($\mathcal{P}(G) \Rightarrow \mathcal{P}(H)$ means if G satisfies property $\mathcal{P}$ then H satisfies $\mathcal{P}$.)

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A *proper s-coloring* $f : V(G) \to \{0, \ldots, s-1\}$ of G is a mapping such that $f(u) \neq f(v)$ if $uv \in E(G)$. The *chromatic number* $\chi(G)$ of a graph G is the smallest s such that G has a proper s-coloring.

s-Partite

- *G is s-partite $\Leftrightarrow$ $\mathsf{TS}(G)$ is s-partite.* In other words, $\chi(G) = \chi(\mathsf{TS}(G)) \geq \chi(\mathsf{TS}_k(G))$.
  - (by Masahiro TAKAHASHI.) Let $f : V(G) \to \{0, \ldots, s-1\}$ be a proper s-coloring of G. Then $g : V(\mathsf{TS}(G)) \to \{0, \ldots, s-1\}$ defined by $g(I) = \sum_{v \in I} f(v) \mod s$ is a proper s-coloring of $\mathsf{TS}(G)$.

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- *There exists a graph G such that $\chi(\mathsf{TS}_k(G)) < \chi(G)$.*
  - Take a graph G having a vertex v that is adjacent to all other vertices and let $G' = G - v$. We have $\chi(G) = \chi(G') + 1 \geq \chi(\mathsf{TS}_k(G')) + 1 = \chi(\mathsf{TS}_k(G)) + 1$.

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$
Eulerian

A graph G is *Eulerian* $\Leftrightarrow$ G is connected and all vertices of G have even degree.

Eulerian

- $\mathsf{TS}_k(C_n)$ *is Eulerian*, for $1 \leq k < n/2$.
  - For $1 \leq k < n/2$, $\mathsf{TS}_k(C_n)$ is connected. Take any $I \in \mathsf{TS}_k(C_n)$. Only maximal paths $P = v_1 v_2 \dots v_{2i+1}$ in C_n satisfying $\{v_1, v_3, \dots, v_{2i+1}\} \subseteq I$ affect the degree of I. (Each path contributes 0 or 2.)

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- For any *Eulerian graph G* on $n \geq 4$ vertices, *every component of $\mathsf{TS}_2(G)$ is Eulerian*.
  - Every vertex of $\mathsf{TS}_2(G)$ has even degree.

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- For any *Eulerian graph* G on $n \geq 4$ vertices, *every component of* $\mathsf{TS}_2(G)$ *is Eulerian*.
  - Every vertex of $\mathsf{TS}_2(G)$ has even degree.
- There exists an *Eulerian graph* G where $\mathsf{TS}_k(G)$ *is not Eulerian*, for any $k \in \{3, \dots, \alpha(G)\}$.
- For any graph G, if $\mathsf{TS}(G)$ is Eulerian, so is G. Moreover, for any $k \geq 2$, one can construct a graph G such that G *is not Eulerian but* $\mathsf{TS}_k(G)$ *is*.

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$
Eulerian

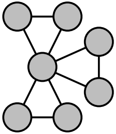
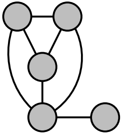
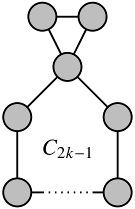
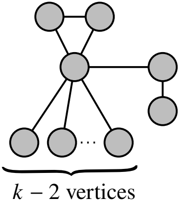
k	(a) G	(a) $\mathsf{TS}_k(G)$	(b) G	(b) $\mathsf{TS}_k(G)$
$k = 2$		disconnected		C_3
$k \geq 3$		has degree-3 vertex		C_4

Figure: (a) G is Eulerian and $\mathsf{TS}_k(G)$ is not, (b) G is not Eulerian and $\mathsf{TS}_k(G)$ is.

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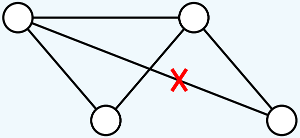
The Clique's Size

- *G has a K_n $\Leftrightarrow$ $\mathsf{TS}(G)$ has a K_n $(n \geq 3)$.*
  - If $\mathsf{TS}_k(G)$ has a K_3, so does G.

$I_1 = \{a_1, a_2, \ldots, a_k\}$ $\quad$ $I_2 = \{x, a_2, \ldots, a_k\}$



$I_3 = \{y, a_2, \ldots, a_k\}$ $\quad$ $I_3 = \{x, z, \ldots, a_k\}$

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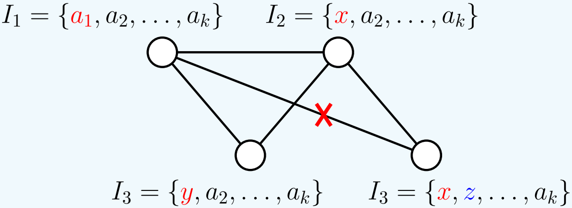
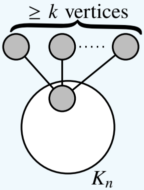
The Clique's Size

- *G has a K_n* $\Leftrightarrow$ $\mathsf{TS}(G)$ *has a K_n ($n \geq 3$)*.
  - If $\mathsf{TS}_k(G)$ has a K_3, so does G.



- There exists a graph G s.t. *G has a K_n and* $\mathsf{TS}_k(G)$ *($k \geq 2$) does not*.



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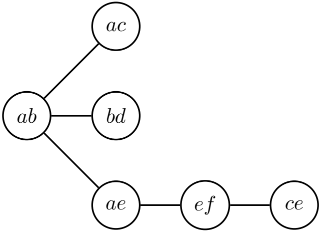
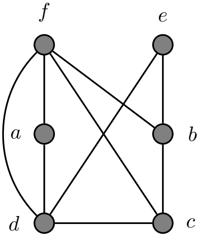
Open Question

A forest T is a TS_k-graph $\Leftrightarrow$?

Remark: "Being a TS_k-graph" is not hereditary even for trees. For example, *$K_{1,3}$ is not a TS_2-graph*, but *the graph obtained by replacing an edge of $K_{1,3}$ by a P_4 is*.



$T \simeq \mathsf{TS}_2(G)$

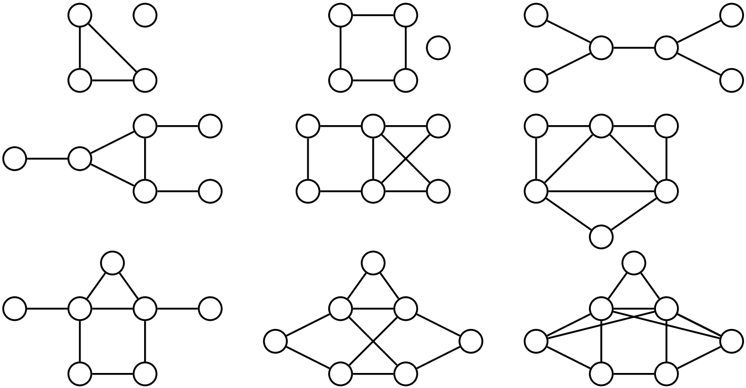


G

Open Questions
Can Forests Be TS_k-Graphs?

We have constructed a (complete?) list $\mathcal{G}$ of graphs such that *if $\mathsf{TS}_2(G)$ is a forest, G must not contain any member of $\mathcal{G}$ as an induced subgraph*. (Thanks to Jesper JANSSON for helpful discussions.)

- $\overline{C_n}$ $(n \geq 4)$,
- and the following graphs:



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Note

If $\mathsf{TS}_2(G)$ *is a forest*, G is a *weakly chordal graph* ($\subseteq$ perfect graph).

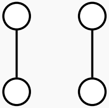
- G is $\overline{C_4}$-free $\Leftrightarrow$ G is $2K_2$-free $\Rightarrow$ G is C_n-free ($n \geq 6$).
- G is $\overline{C_n}$-free ($n \geq 4$) $\Rightarrow$ G is $(C_n, \overline{C_n})$-free ($n \geq 5$) $\Leftrightarrow$ G is weakly chordal ($\subseteq$ perfect).

Open Questions

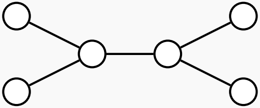
Can Forests Be TS_k-Graphs?

Proposition 7

Let F be a forest. $\mathsf{TS}_2(F)$ *is a forest* $\Leftrightarrow$ F *is* $(2K_2, H)$*-free.*



$2K_2 \simeq \overline{C_4}$



H-graph

Consequently,

Let F be a tree. $\mathsf{TS}_2(F)$ *is a tree* $\Leftrightarrow$ $F \simeq P_n$ *($1 \leq n \leq 4$).*

($\Rightarrow$) $\mathsf{TS}_2(2K_2) \simeq C_4$ and $\mathsf{TS}_2(H) \simeq C_8 + 2K_1$.

($\Leftarrow$) If F has a cycle and is $2K_2$-free, it contains an induced H-graph.

Question

Let G be a non-acyclic graph. Then $\mathsf{TS}_2(G)$ is a forest $\Leftrightarrow$?

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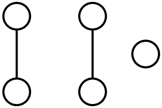
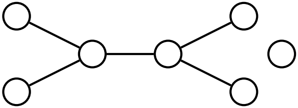
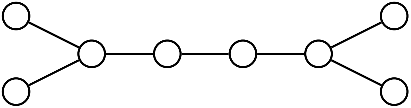
- Let F be a forest. *If $\mathsf{TS}_3(F)$ is a forest, F does not contain any of the following graphs as an induced subgraph*



$2K_2 + K_1$



$H + K_1$



H'

Question

If $\mathsf{TS}_3(F)$ has a cycle, does F contain one of the above graphs as an induced subgraph?

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Part I

Appendix

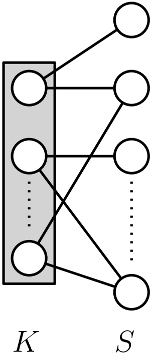

Figure: A split graph $G = (K \cup S, E)$.

Recall

Every split graph G has a KS-partition such that $|K| = \omega(G)$—the maximum size of a clique of G. We denote it by $G = (K \cup S, E)_{K\text{-max}}$.

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Connected Split Graphs

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Planar
Girth

Proposition 8

A connected $G = (K \cup S, E)_{K\text{-max}}$ is a TS$_k$*-graph* $\Leftrightarrow$ *every $v \in K$ has at most $k-1$ neighbors in S and every $w \in S$ has exactly one neighbor in K.*

($\Leftarrow$) We construct a graph H such that $G = \text{TS}_k(H)$.

($\Rightarrow$) How do we label G here?

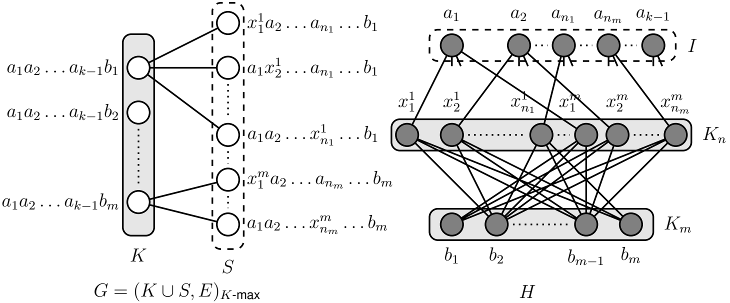


Figure: Construction of a graph H such that $G \simeq \text{TS}_k(H)$.

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$

Planar

Planar

- If G is non-planar, so is $\mathsf{TS}(G)$. ($G \simeq \mathsf{TS}_1(G)$)

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$
Planar

Planar

- If G is non-planar, so is $\mathsf{TS}(G)$. ($G \simeq \mathsf{TS}_1(G)$)
- [Computer search] With respect to the number of vertices, C_7 is the unique smallest planar graph G for which $\mathsf{TS}(G)$ is non-planar.
  - For each of 99 connected planar graphs G on 6 vertices, $\mathsf{TS}(G)$ is planar.
  - $\mathsf{TS}_2(C_7)$ is non-planar.

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$
Planar

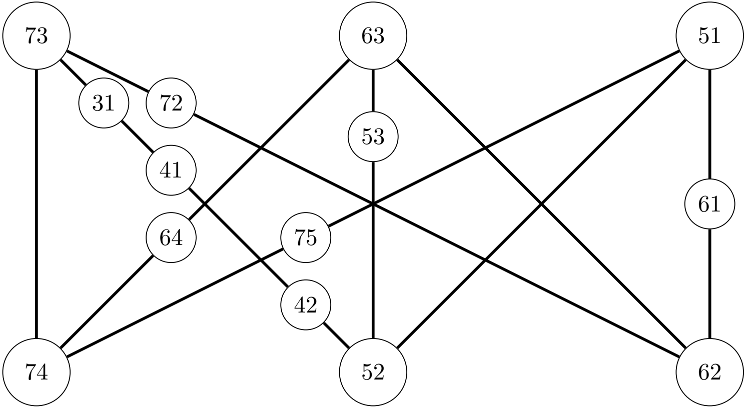
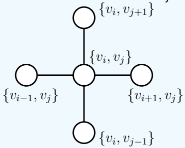


Figure: A subdivision of $K_{3,3}$ that is contained in $\mathsf{TS}_2(C_7)$. Each number of the form ab inside a node represents an independent set $\{v_a, v_b\}$ of $C_7 = v_1 \dots v_7 v_1$.

Planar

- If G is non-planar, so is TS(G). ($G \simeq \mathsf{TS}_1(G)$)
- [Computer search] With respect to the number of vertices, C_7 is the unique smallest planar graph G for which TS(G) is non-planar.
  - For each of 99 connected planar graphs G on 6 vertices, TS(G) is planar.
  - $\mathsf{TS}_2(C_7)$ is non-planar.
- $\mathsf{TS}_2(P_n)$ is planar for every $n \geq 3$.
  - $\mathsf{TS}_2(P_n)$ is a subgraph of a $n \times n$ grid.

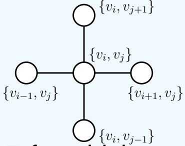


Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$

Planar

Planar

- If G is non-planar, so is $\mathsf{TS}(G)$. ($G \simeq \mathsf{TS}_1(G)$)
- [Computer search] With respect to the number of vertices, C_7 is the unique smallest planar graph G for which $\mathsf{TS}(G)$ is non-planar.
  - For each of 99 connected planar graphs G on 6 vertices, $\mathsf{TS}(G)$ is planar.
  - $\mathsf{TS}_2(C_7)$ is non-planar.
- $\mathsf{TS}_2(P_n)$ is planar for every $n \geq 3$.
  - $\mathsf{TS}_2(P_n)$ is a subgraph of a $n \times n$ grid.



- [Computer search] P_8 is the largest path P for which $\mathsf{TS}(P)$ is planar.
  - $\mathsf{TS}(P_8)$ is planar, and $\mathsf{TS}_3(P_9)$ is not.

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$

Planar

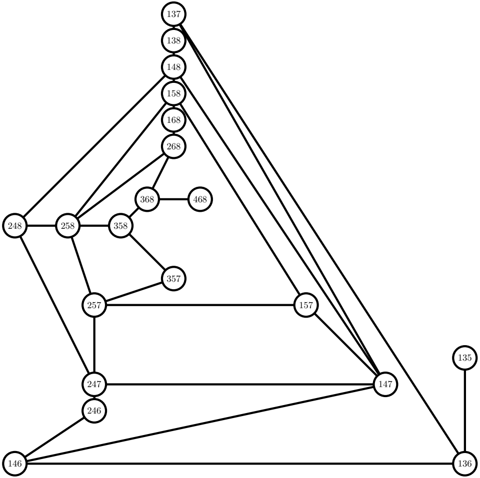


Figure: A planar drawing of $\mathsf{TS}_3(P_8)$. Each number of the form abc inside a node represents an independent set $\{v_a, v_b, v_c\}$ of $P_8 = v_1 \dots v_8$.

Graph Properties: G vs. $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$

Planar

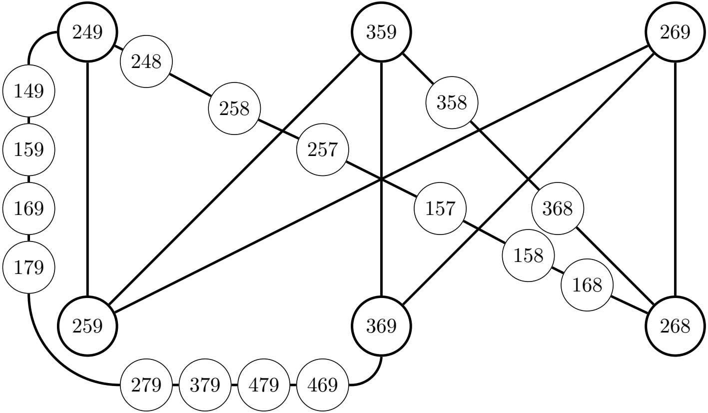


Figure: A subdivision of $K_{3,3}$ that is contained in $\mathsf{TS}_3(P_9)$. Each number of the form abc inside a node represents an independent set $\{v_a, v_b, v_c\}$ of $P_9 = v_1 \dots v_9$.

The *girth* of a graph G is the smallest size of a cycle in G, and is ∞ if G is acyclic.

Girth

- girth$(\mathsf{TS}(P_n))$ is 4 for every $n \geq 5$ and ∞ otherwise.
  - TS(P_n) has a C_4 when $n \geq 5$. Moreover, since P_n is bipartite, so is TS(P_n), i.e., it does not contain a C_3.

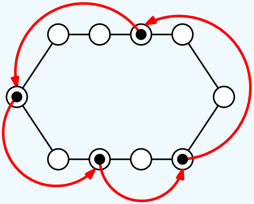
Graph Properties: G vs. TS$_k(G)$ and TS(G)

Girth

The *girth* of a graph G is the smallest size of a cycle in G, and is ∞ if G is acyclic.

Girth

- girth(TS(P_n)) is 4 for every $n \geq 5$ and ∞ otherwise.
  - TS(P_n) has a C_4 when $n \geq 5$. Moreover, since P_n is bipartite, so is TS(P_n), i.e., it does not contain a C_3.
- For $1 \leq k < n/2$, girth(TS$_k(C_n)$) $= n$. If $k = n/2$, we have girth(TS$_k(C_n)$) $= \infty$.
  - Let $I \in$ TS$_k(G)$. How to form a cycle in TS$_k(C_n)$ containing I?



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