Open Problems

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1. Token Sliding Graphs
2. Distance-d Independent Set Reconfiguration
3. k-Path Vertex Cover Reconfiguration

Some Definitions

- Two vertex subsets I and J of a graph G are *adjacent under Token Jumping (TJ)* if there exist $u, v \in V(G)$ such that $I \setminus J = \{u\}$ and $J \setminus I = \{v\}$. If $uv \in E(G)$, we say that I and J are *adjacent under Token Sliding (TS)*.

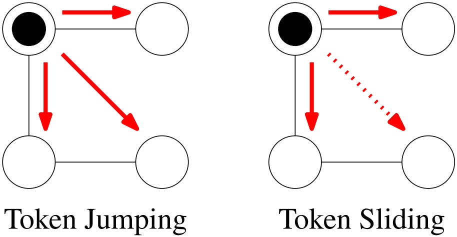

Figure: An example of TS and TJ.

- An *independent set (IS)* of G is a vertex subset $I \subseteq V(G)$ such that for each $u, v \in I$, we have $uv \notin E(G)$.
- A *double-broom graph* D_{n_1,n,n_2} is the graph obtained by attaching n_1 leaves at one endpoint of P_n and n_2 leaves at the other endpoint.

Token Sliding Graphs

Given a graph G and a positive integer k.

- The TS_k*-graph of* G, denoted by $\mathsf{TS}_k(G)$, is the graph whose nodes are independent sets of G and edges are defined under Token Sliding (TS).

Avis and Hoang 2022

$\mathsf{TS}_k(G)$ is a forest $\Leftrightarrow$ G satisfies?

- The question remains open even when G *is a forest*.
- **Known:** If G is a forest, $\mathsf{TS}_2(G)$ is a forest $\Leftrightarrow$ G has no induced $2K_2$ and $D_{2,2,2}$.
- **Conjecture:** If G is a forest, $\mathsf{TS}_k(G)$ ($k \geq 3$) is a forest $\Leftrightarrow$ G has no induced $2K_2 + (k-2)K_1$, $D_{2,2,2} + (k-2)K_1$, and $D_{2,4,2} + (k-3)K_1$.

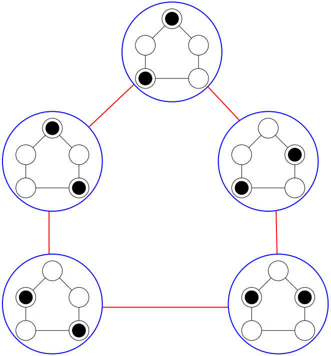

Figure: $\mathsf{TS}_2(C_5)$.

Token Sliding Graphs

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Avis and Hoang 2022

Given a graph G, what are the necessary and sufficient conditions for G to be a TS_k-graph (of some graph H)?

- *Solved* for complete graphs, paths, cycles, complete bipartite graphs, and connected split graphs.
- How about *other graphs*?

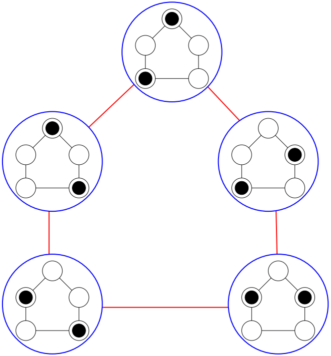

Figure: $\mathsf{TS}_2(C_5)$.

Distance-d Independent Set Reconfiguration

A *distance-d independent set (DdIS)* of G a vertex subset $I \subseteq V(G)$ such that *for each $u, v \in I$, their distance (i.e., the length of a shortest uv-path) in G is at least d*, for some given positive integer $d \geq 2$. I is a D2IS if and only if I is an IS.

DISTANCE-d INDEPENDENT SET RECONFIGURATION (DdISR) under R ($d \geq 2$)
Input: (G, I, J, R) where I, J are DdISs of G and $\mathsf{R} \in \{\mathsf{TS}, \mathsf{TJ}\}$
Question: Does there exist a sequence of adjacent DdISs under R between I and J?

It has been shown in [Hoang 2022a] that:

- On chordal graphs, under TJ, DdISR is in P when d is even and PSPACE-complete when d is odd.
- On split graphs, under TS, DdISR is PSPACE-complete when $d = 2$ and in P when $d \geq 3$, while under TJ, it is in P when $d \neq 3$ and is PSPACE-complete when $d = 3$.
- DdISR remains PSPACE-complete for $d \geq 3$ on general graphs, perfect graphs, and planar graphs of maximum degree 3 and bounded bandwidth.

Distance-d Independent Set Reconfiguration

Hoang 2022a

Is DdISR under TS on trees in P for $d \geq 3$?

Let $(T, I, J, \mathsf{TS}, d)$ be a DdISR's instance under TS on a tree T. Suppose that for every token t on some $u \in I \cup J$, there is a TS-sequence in T that slides t to one of u's neighbors, i.e., *every token can be moved out of its original position*.

- For $d = 2$, it is *always a yes-instance*, and the problem can be solved in linear time [Demaine et al. 2015].
- For $d \geq 3$, there *exists a no-instance*.

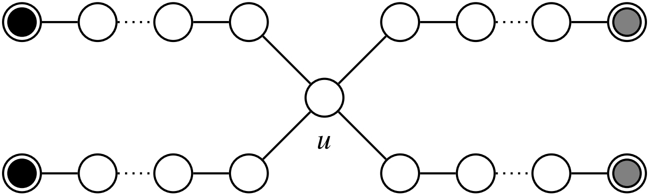

Figure: A no-instance $(T, I, J, \mathsf{TS}, d)$ $(d \geq 3)$ where every token is "movable". Tokens in I (resp., J) are marked with the black (resp. gray) color. All tokens are of distance $d - 1$ from u.

k-Path Vertex Cover Reconfiguration

A *k-path vertex cover (k-PVC)* of G is a vertex subset $I \subseteq V(G)$ such that *each path on k vertices contains at least one vertex in I*, for some given positive integer $k \geq 2$. I is a 2-PVC if and only if $V(G) \setminus I$ is an IS.

k-PATH VERTEX COVER RECONFIGURATION (k-PVCR) under R **Input:** (G, I, J, R) where I, J are k-PVCs of G and $\mathsf{R} \in \{\mathsf{TS}, \mathsf{TJ}\}$ **Question:** Does there exist a sequence of adjacent k-PVCs under R between I and J?

Hoang, Suzuki, and Yagita 2022

What is the complexity of k-PVCR

- under TS/TJ on bipartite graphs?
- under TS on trees?

- k-PVCR under TJ on trees is in P [Hoang, Suzuki, and Yagita 2022].
- The second question has been *partially answered* in [Hoang 2022b] on caterpillars (i.e., trees obtained by attaching leaves to a path) for $k \geq 4$.

References

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