On Reconfiguration Graph of Independent Sets under Token Sliding

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TS-Reconfiguration Graph of Independent Sets

Given a graph $G = (V, E)$ and a positive integer k.

	G	*Token Graph* $F_k(G)$	TS$_k$-*(Reconfiguration) Graph* TS$_k(G)$
Vertex	$V(G)$	k-Vertex Subsets of $V(G)$	*Independent* k-Vertex Subsets of $V(G)$
Edge	$E(G)$	Token Sliding	Token Sliding
		$F_2(G)$ [Alavi, Behzad, Erdős, and Lick 1991]	[This Poster]

- Each vertex of G contains at most *one unlabeled* token.
- *Token Sliding (TS)* involves moving a token from one vertex to one of its *unoccupied adjacent* vertices.

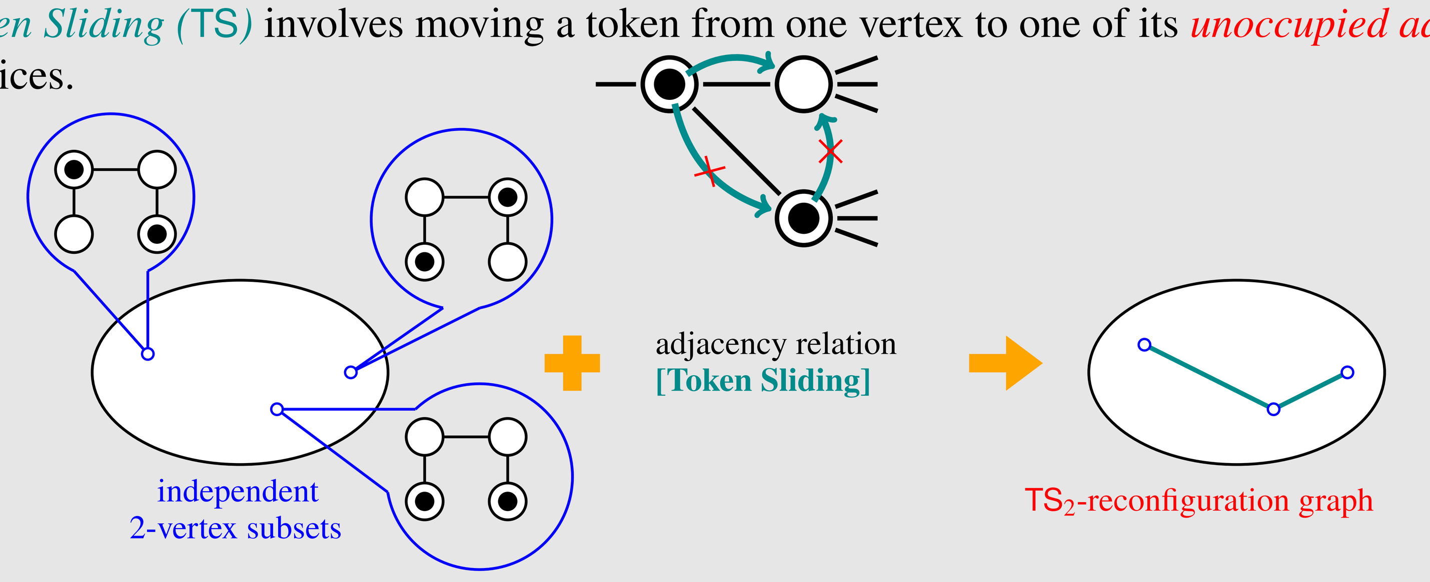


Figure 1. TS$_2(P_4) \subseteq F_2(P_4)$

We focus on two graphs:

- TS$_k(G)$ whose nodes are independent sets of *size* k, and
- TS(G) whose nodes are independent sets of *arbitrary size*.

Motivation

Most of the known research on TS$_k(G)$ have been focused on *designing efficient algorithms* and *showing computational hardness* of several *reconfiguration questions* [Nishimura 2018].

- Reachability/Shortest Transformation: Is there a (shortest) path between two given nodes of TS$_k(G)$?
- and so on.

We look at TS$_k(G)$ from *a purely graph-theoretic viewpoint*.

(Q1) *Is G a TS$_k$-reconfiguration graph ($k \geq 2$)*, i.e., does there exist H such that $G \simeq$ TS$_k(H)$?

(Q2) *If G satisfies some property $\mathcal{P}$, does TS(G)/TS$_k(G)$ also satisfy $\mathcal{P}$, and vice versa?*

(Q1) Given G and $k \geq 2$, is G a TS$_k$-graph?

Line Graph $L(G)$ and TS$_2$-Graph TS$_2(\overline{G})$

- TS$_2(\overline{G})$ is a *(spanning) subgraph* of $L(G)$.
- TS$_2(\overline{G}) \simeq L(G)$ if and only if G is *triangle-free*.

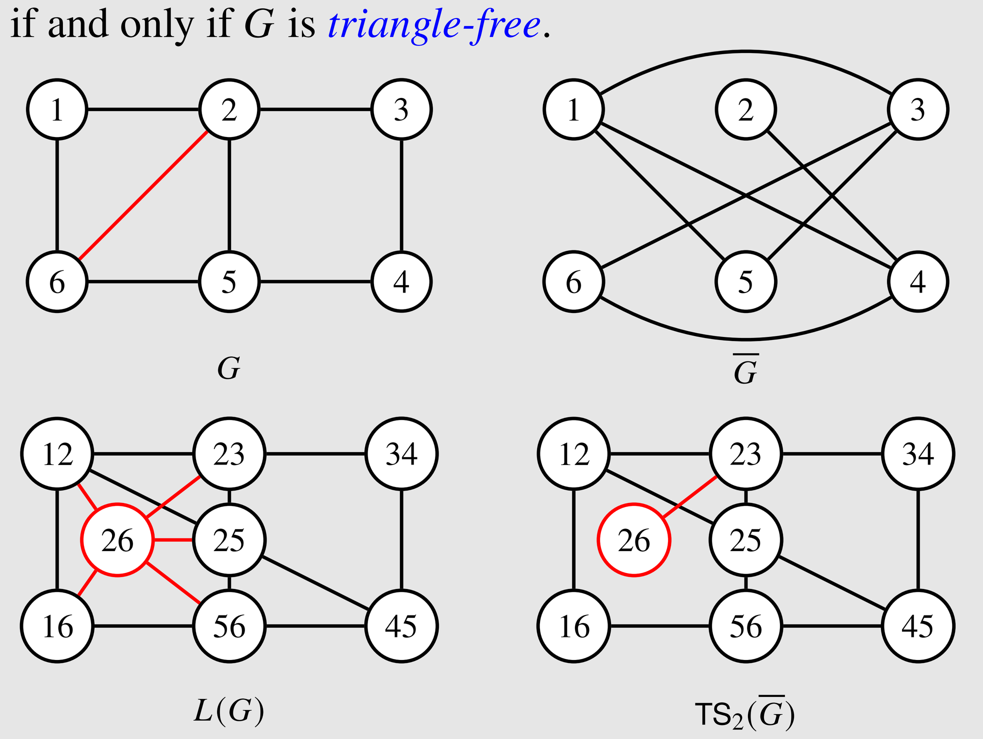


Figure 2. $L(G)$ v.s. TS$_2(\overline{G})$.

Complete Graph K_n, Path P_n, and Cycle C_n Are TS$_k$-Graphs

- Given H and let *G = TS$_{\alpha(H)}(H)$*. Then, *for every $k \geq \alpha(H)$, G is a TS$_k$-graph.*

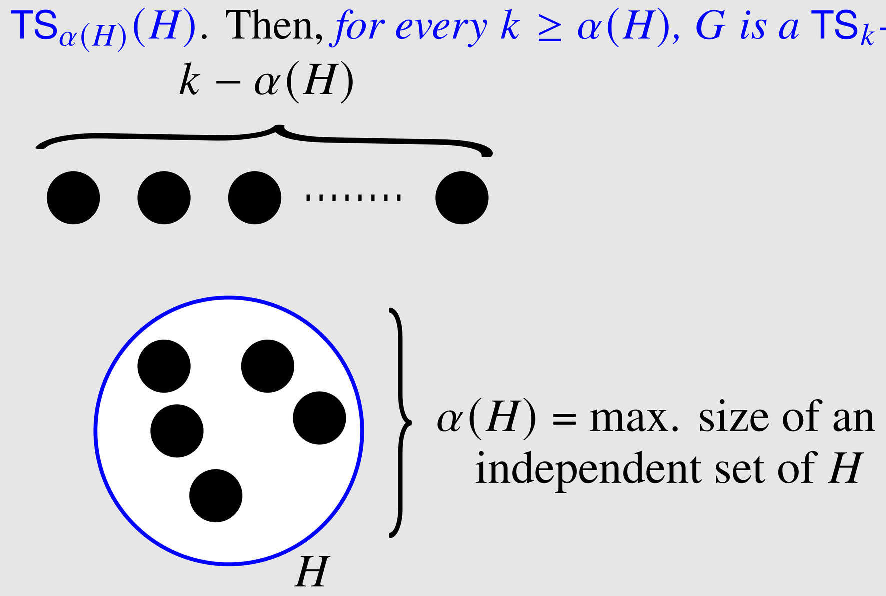


Figure 3. A graph H' s.t. G = TS$_{\alpha(H)}(H) \simeq$ TS$_k(H')$ for $k \geq \alpha(H)$.

- Since $\alpha(K_n) = 1$, the graph *K_n = TS$_1(K_n)$ is a TS$_k$-graph for all $k \geq \alpha(K_n) = 1$.*
- For $k = 2$, we have $P_n = L(P_{n+1}) \simeq$ TS$_2(\overline{P_{n+1}})$. For $k \geq 3$, since $\alpha(\overline{P_{n+1}}) = 2$, the graph *$P_n$ is a TS$_k$-graph for all $k \geq \alpha(\overline{P_{n+1}}) = 2$.*
- $C_3 \simeq K_3$ is clearly a TS$_k$-graph. For C_n ($n \geq 4$), apply similar arguments as in the case of P_n.

Complete Bipartite Graph $K_{m,n}$ ($m \leq n$) Is a TS$_k$-Graph ⇔ Either $m = 1$ and $n \leq k$ or $m = n = 2$

(⇐) The case $m = n = 2$ is clear, since $K_{2,2} \simeq C_4$. For $m = 1$ and $n \leq k$, we construct a graph G such that $K_{m,n} \simeq$ TS$_k(G)$.

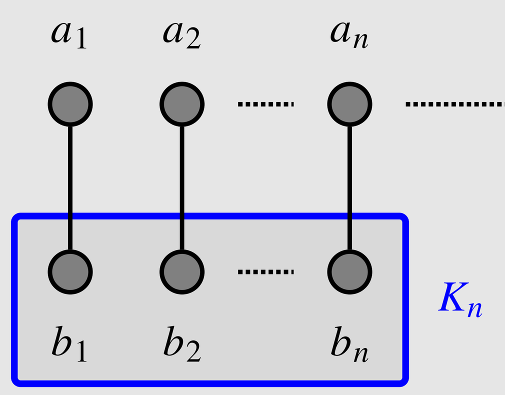


Figure 4. A graph G such that $K_{1,n} \simeq$ TS$_k(G)$ where $n \leq k$.

(⇒) Suppose to the contrary that there exists G such that $K_{m,n} \simeq$ TS$_k(G)$, where either (a) $m = 1$ and $n \geq k + 1$ or (b) $m = 2$ and $n > 2$ ($m \leq n$). *Can we label vertices of $K_{m,n}$ by independent sets of G in each case?*

(Q2) If G satisfies $\mathcal{P}$, does TS(G)/TS$_k(G)$ also satisfy $\mathcal{P}$, and vice versa?

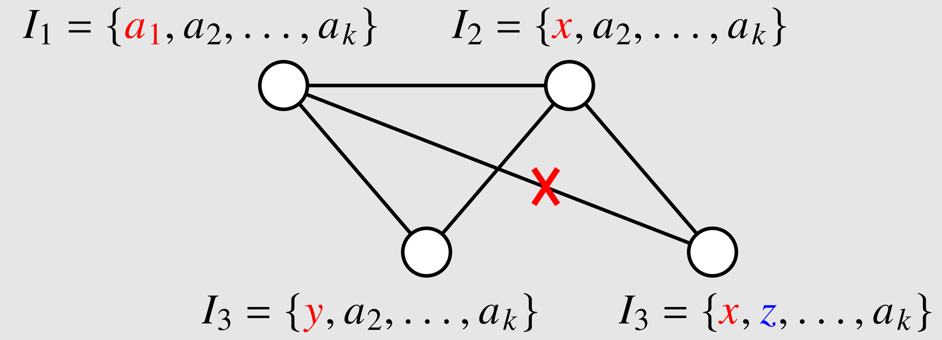
Our Answers For Selected Properties $\mathcal{P}$

$\mathcal{P}$	G	$\mathcal{P}(G) \Rightarrow \mathcal{P}(\text{TS}(G))$	$\mathcal{P}(\text{TS}(G)) \Rightarrow \mathcal{P}(G)$	$\mathcal{P}(G) \Rightarrow \mathcal{P}(\text{TS}_k(G))$	$\mathcal{P}(\text{TS}_k(G)) \Rightarrow \mathcal{P}(G)$
s-partite	general	yes			no
planar	P_n	yes, iff $n \leq 8$		yes, iff $k = 2, n \geq 3$ or $k \geq 3, n \leq 8$	
	tree	yes, iff $n \leq 7$			
	C_n	yes, iff $n \leq 6$			
	connected				
Eulerian	C_n	no	yes	yes, iff $1 \leq k < n/2$	
	general	no	yes	no	no
acyclic	P_n	yes, iff $n \leq 4$			
non-acyclic	C_n	no	yes, iff $1 \leq k < n/2$		
having K_s	general	yes		no	yes

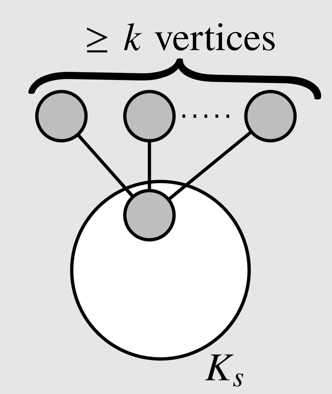
Table 1. Some properties of (reconfiguration) graphs. Here $n = |V(G)|$. ($\mathcal{P}(G) \Rightarrow \mathcal{P}(H)$ means if G satisfies property $\mathcal{P}$ then H satisfies $\mathcal{P}$.)

The Clique's Size

- G has a K_s ⇔ TS(G) has a K_s ($s \geq 3$). The key observation is: *If TS(G) has a K_3, so does G.*



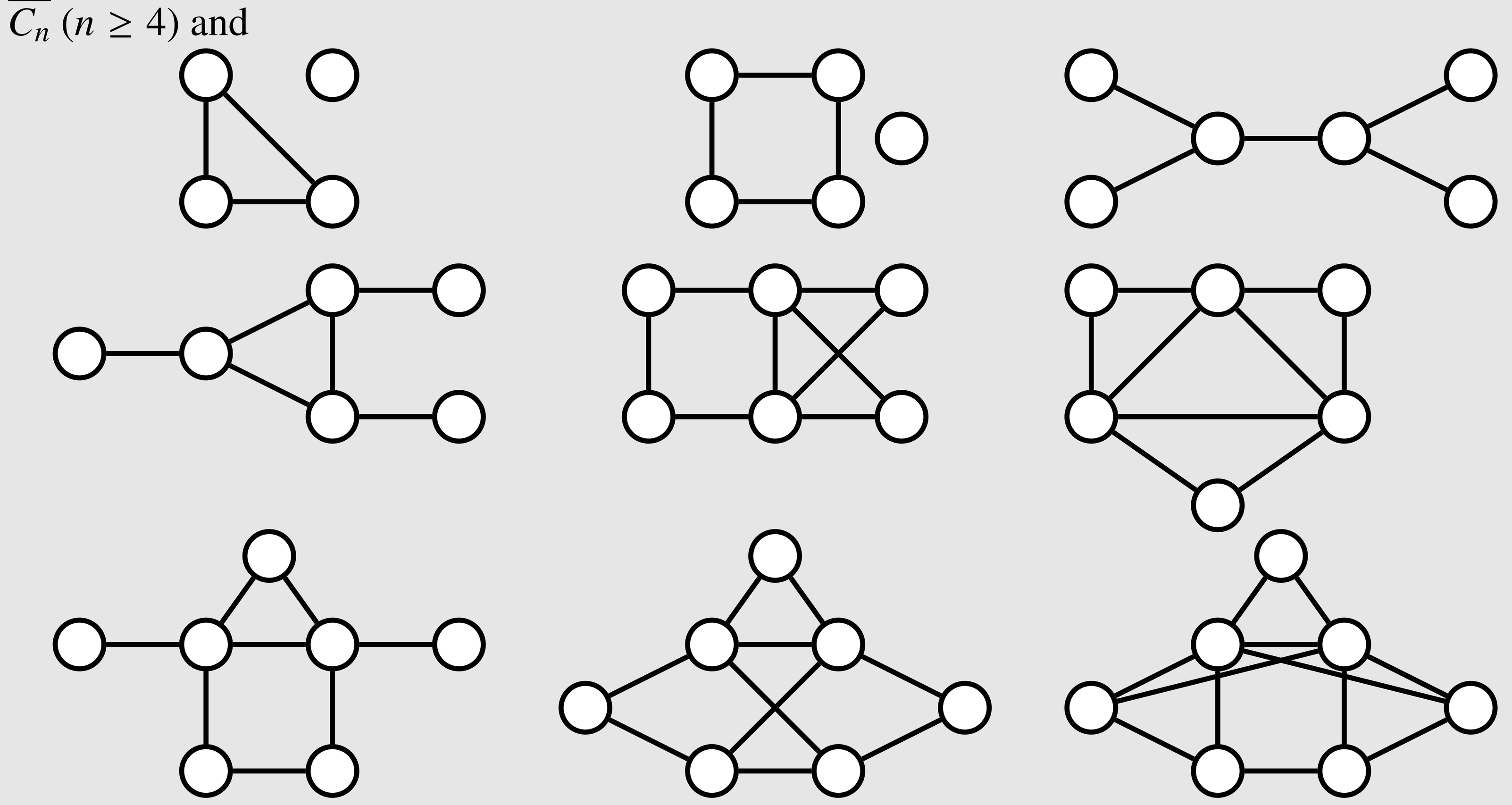
- There exists a graph G such that *G has a K_s and TS$_k(G)$ ($k \geq 2$) does not.*



(Open Question) Given G and $k \geq 2$, the graph TS$_k(G)$ is acyclic ⇔ ?

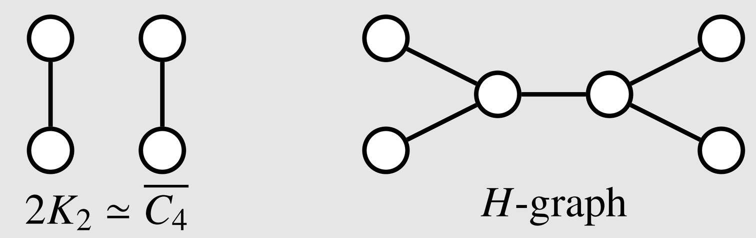
Some Results and Questions for TS$_2(G)$

- *If TS$_2(G)$ is acyclic*, G must not contain any of the following graphs as an induced subgraph: $\overline{C_n}$ ($n \geq 4$) and



Consequently, G must be a *weakly chordal graph*, i.e., G is $(C_n, \overline{C_n})$-free for $n \geq 5$. ($\overline{C_4}$-free ⇒ C_n-free for $n \geq 6$.)

- ***Question:** If TS$_2(G)$ has a cycle, does it contain one of the above graphs as an induced subgraph?*
- *Given a forest G, the graph TS$_2(G)$ is a forest ⇔ G is $(2K_2, H)$-free.*



(⇒) TS$_2(2K_2) \simeq C_4$ and TS$_2(H) \simeq C_8 + 2K_1$.

(⇐) If G has a cycle and is $2K_2$-free, it contains an induced H-graph.

References

- Nishimura, N. (2018). "Introduction to Reconfiguration". In: *Algorithms* 11.4, p. 52. doi: 10.3390/a11040052.
- Alavi, Y., M. Behzad, P. Erdős, and D. R. Lick (1991). "Double Vertex Graphs". In: *Journal of Combinatorics, Information & System Sciences* 16.1, pp. 37–50.