Sliding Token on Bipartite Permutation Graphs

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Reconfiguration Problems

[Flake & Baum 2002]

[Romanishin, Rus, Gilpin 2013]

[Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, Uno 2008]


NIKKEI
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TOHO

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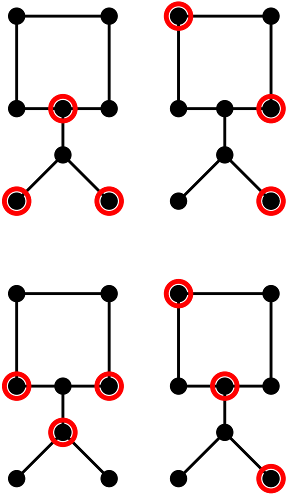
- Question: is there a sequence of transformations between two given solutions? (PSPACE-complete for Rush Hour)

Sliding Token

Sliding Token:
a natural, pure problem in Combinatorial Reconfiguration

Sliding Token



- Classic optimization problem: **Independent Set**

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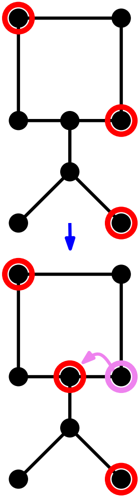

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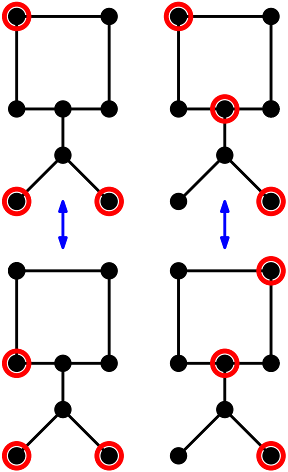

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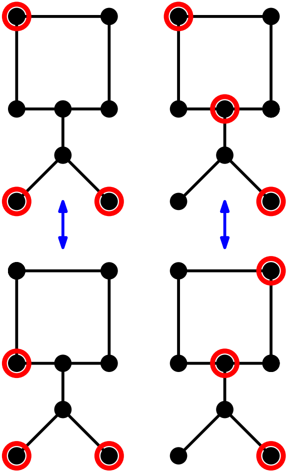

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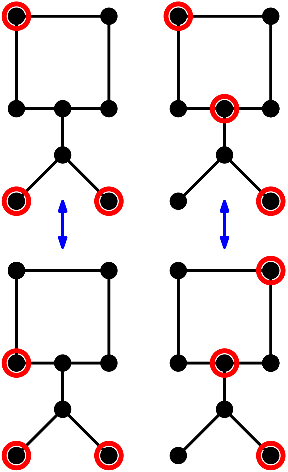

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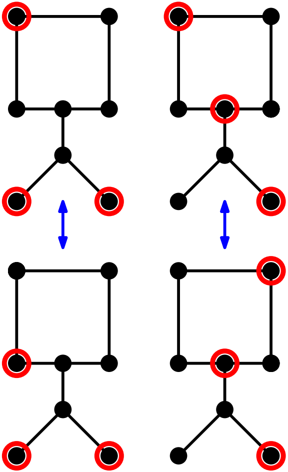

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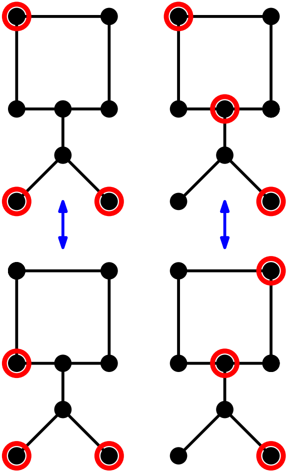

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  - Notation: $[A]$ is A's connected component
  - Ask: $B \in [A]$?

A Brief Overview of SLIDING TOKEN's Complexity

- PSPACE-complete on general, AT-free, planar, perfect, and bounded treewidth graphs [Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, Uno 2008], [Wrochna 2014]
- Polytime on proper interval graphs, claw-free graphs, forests, cographs [Bonsma, Kamiński, Wronchna 2014], [Demaine, Demaine, F., Hoang, Ito, Ono, Otachi, Uehara, Yamada 2014], [Kamiński, Medvedev, Milanic 2010]
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- **??? on bipartite graphs**
  - We give an efficient algorithm on a subclass of bipartite graphs.

Main Result

Algorithm for SLIDING TOKEN on **bipartite permutation** graphs.

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Algorithm for Sliding Token on **bipartite permutation** graphs.

Given graph G, independent sets A and B,
finds a reconfiguration sequence from A to B
or reports that none exists.

Bipartite Permutation Graphs

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(Sliding Token is PSPACE-hard on AT-free graphs)

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Bipartite permutation graph iff

vertices can be ordered $v_1, v_2, \ldots, v_n$

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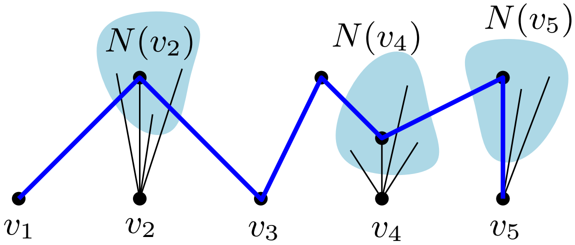

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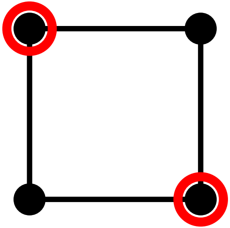

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4. **Canonical representatives** of reconfiguration graph's connected components

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- $B \in [A]$ iff $A_+ = B_+$.

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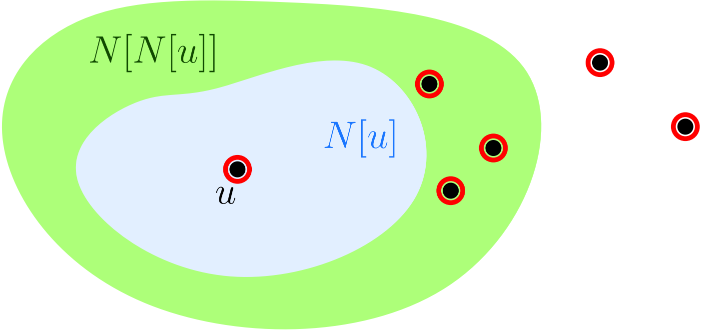
- Repeat, pretending we deleted the token and neighborhood

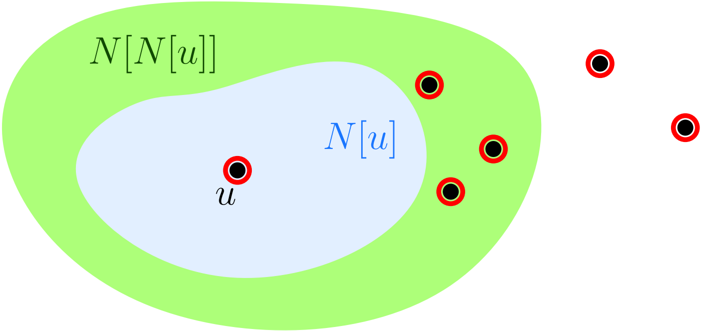
Proof sketch of idea (b): $R(G \setminus N[u], I \setminus \{u\}) = \emptyset$

u: vertex of least index in A_+



Proof sketch of idea (b): $R(G \setminus N[u], I \setminus \{u\}) = \emptyset$

First, push other tokens away from u (extreme case analysis)



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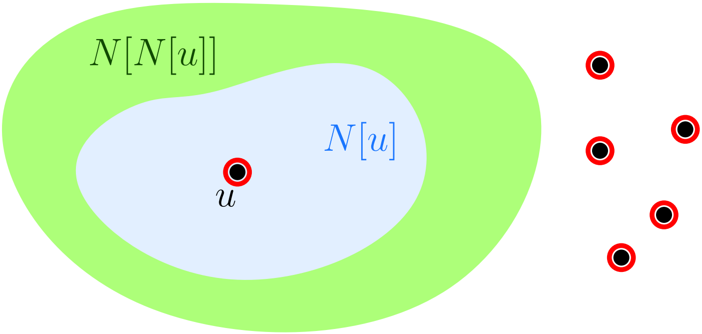
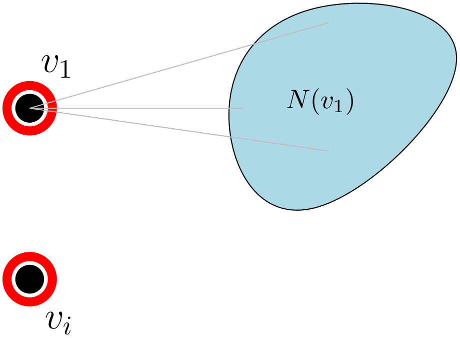


Case: $v_1, v_i \in I$ with $i <$ min index of neighbor of v_1



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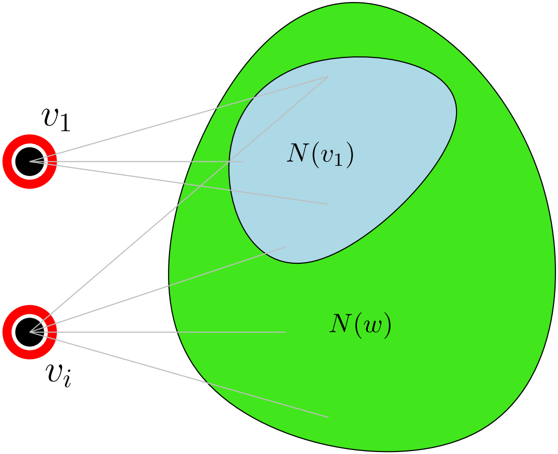

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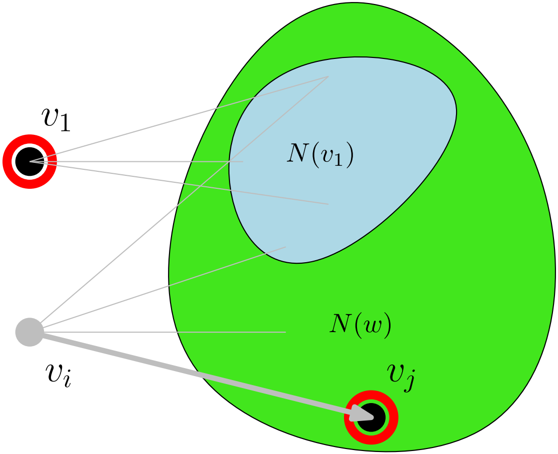

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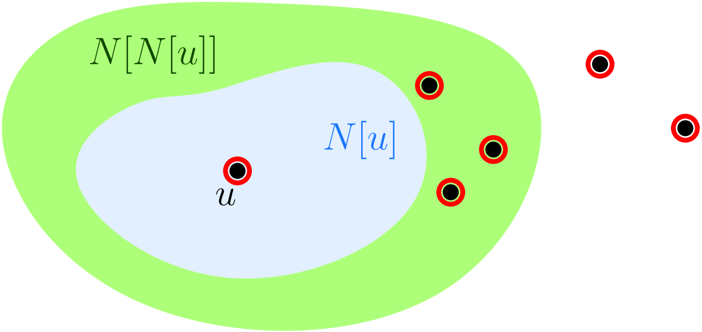

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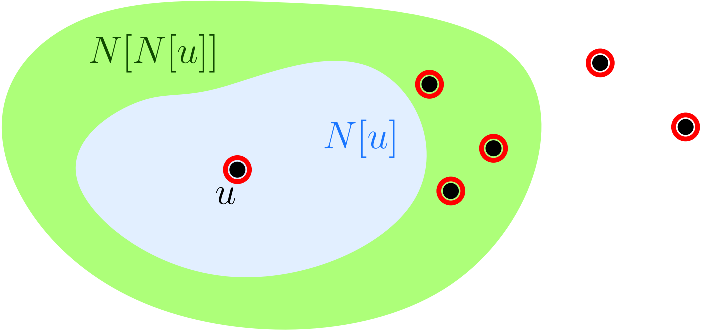

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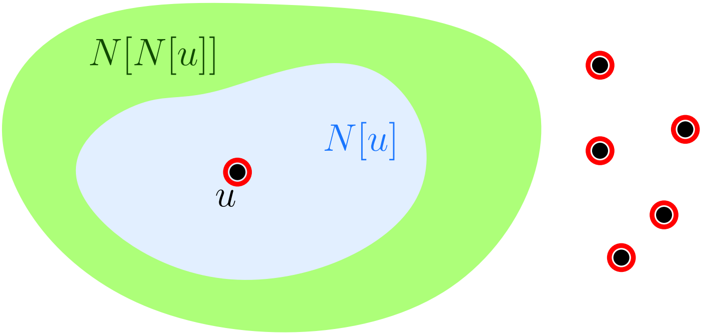

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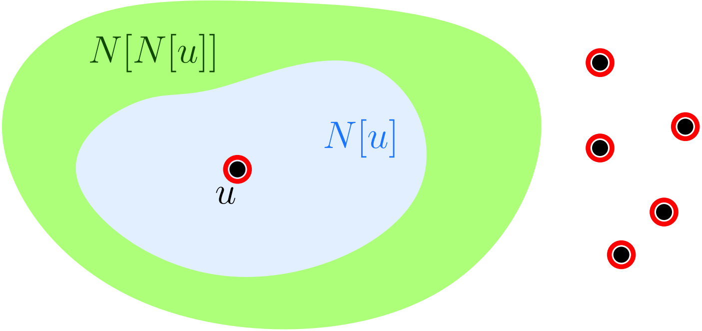

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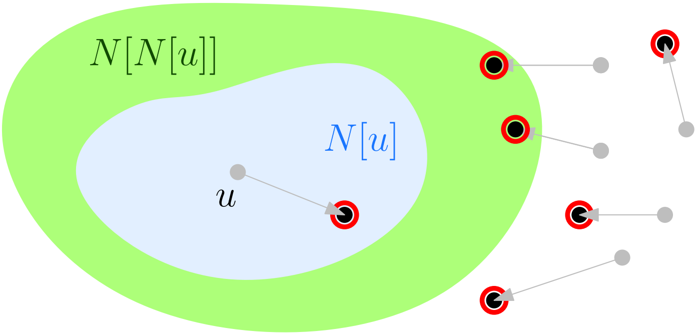
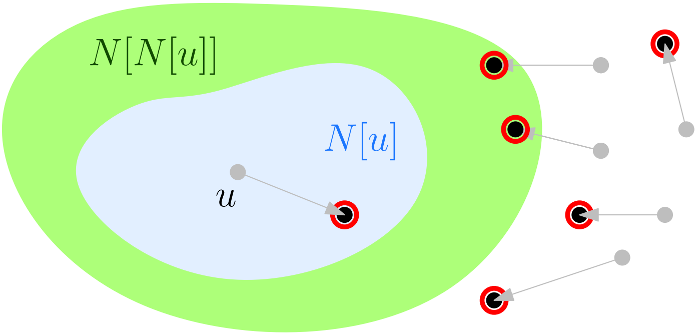


Idea (b) proof:

Now edit sequence so token stays on u



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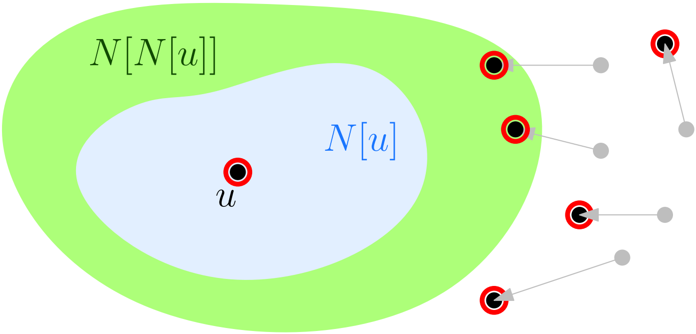
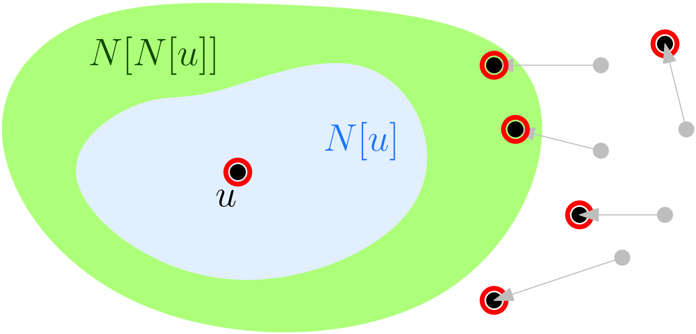


Idea (b) proof:

This sequence witnesses that nothing is rigid after deleting $N[u]$



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Overall, $O(n^3)$ time.

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- Canonical representatives seem hard to generalize: permutation graphs have nice linear structure.
- Cannot naively put a token on some vertex and delete the neighborhood

Thanks

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