A note regarding "Sliding Tokens on a Cactus"

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1 Introduction

In this note, we introduce a counter-example for Lemma 6 of [1] and the progress of resolving this issue. This example was provided by Mathew Francis and Veena Prabhakaran.

2 The problem

Let I, J be two given independent sets of a graph G. Imagine that the vertices of an independent set are viewed as tokens (coins). A token is allowed to move (or slide) from one vertex to one of its neighbors. The SLIDING TOKEN problem asks whether there exists a sequence of independent sets of G starting from I and ending with J such that each intermediate member of the sequence is obtained from the previous one by moving a token according to the allowed rule. If such a sequence exists, we write $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{J}$. In [1], we claimed that this problem is solvable in polynomial time when the input graph is a cactus graph—a graph whose blocks (i.e., maximal 2-connected subgraphs) are cycles.

3 Lemma 6 and its counter-example

Let $\mathbf{I}$ be an independent set of a graph G. Let $W \subseteq V(G)$ and assume that $\mathbf{I} \cap W \neq \emptyset$. We say that a token t placed at some vertex $u \in \mathbf{I} \cap W$ is $(G, \mathbf{I}, W)$-*confined* if for every $\mathbf{J}$ such that $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{J}$, t is always placed at some vertex of W. In other words, t can only be slid along edges of $G[W]$.

Let H be an induced subgraph of G. H is called $(G, \mathbf{I})$-*confined* if $\mathbf{I} \cap H$ is a maximum independent set of H and all tokens in $\mathbf{I} \cap H$ are $(G, \mathbf{I}, V(H))$-confined. In particular, if H is a cycle (resp. a path) of G, we say that it is a $(G, \mathbf{I})$-*confined cycle* (resp. $(G, \mathbf{I})$-*confined path*).

Mathew Francis and Veena Prabhakaran showed us a counter-example of the following proposition

Lemma 3.1 ([1, Lemma 6]). *Let G be a cactus. Let $P = p_1p_2 \dots p_l$ be an induced path in G. Let $\mathbf{I}$ be an independent set of G satisfying that $\mathbf{I} \cap P$ is a maximum independent set of P. Assume that for any $x \in \mathbf{I} \cap P$, the token placed at x is $(G, \mathbf{I})$-movable.*

Then, P is $(G, \mathbf{I})$-confined if and only if l is even (i.e., the length $k = l - 1$ of P is odd) and there exist two independent sets $\mathbf{I}'_1$ and $\mathbf{I}'_2$ such that

(i) $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{I}'$, *where* $\mathbf{I}' \in \{\mathbf{I}, \mathbf{I}'_1, \mathbf{I}'_2\}$,

(ii) $\mathbf{I}'_1 \cap P = \{p_1, p_3, \ldots, p_{l-1}\}$, $\mathbf{I}'_2 \cap P = \{p_2, p_4, \ldots, p_l\}$, *and*

(iii) for every $x \in \mathbf{I}' \cap P$, *the token placed at* x *is* $(G^x_P, \mathbf{I}' \cap G^x_P)$*-rigid.*

Below are the contents of their counter-example.

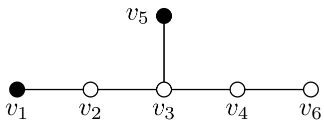
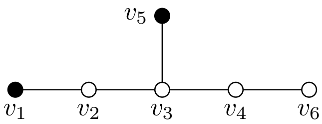


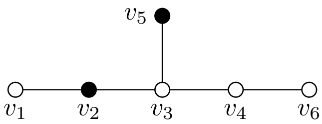
Figure 1: A cactus graph G and an independent set $\mathbf{I}$ (given as black vertices), where $P = v_1v_2$

Consider the cactus graph shown in Figure 1. The black vertices show an independent set $\mathbf{I}$, and we take P to be the path containing just the two vertices v_1 and v_2. Clearly, the token on v_1 is not $(G, \mathbf{I}, V(P))$-confined, since we can first move the token on v_5 to v_6, and then move the token at v_1 to v_3. Thus P is not $(G, \mathbf{I})$-confined.

Let $\mathbf{I'_1} = \mathbf{I}$, and let $\mathbf{I'_2}$ be the independent set obtained from $\mathbf{I}$ by moving the token on v_1 to v_2. These two independent sets are shown in Figure 2(a) and 2(b) respectively.



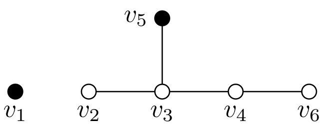
(a) Independent set $\mathbf{I'_1}(= \mathbf{I})$ reachable from $\mathbf{I}$ such that $\mathbf{I'_1} \cap P = \{v_1\}$



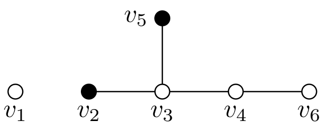
(b) Independent set $\mathbf{I'_2}$ reachable from $\mathbf{I}$ such that $\mathbf{I'_2} \cap P = \{v_2\}$

Figure 2: The two independent sets $\mathbf{I'_1}$ and $\mathbf{I'_2}$

Clearly, the independent sets $\mathbf{I'_1}$ and $\mathbf{I'_2}$ satisfy conditions (i) and (ii) of Lemma 6. As Figure 3 shows, it can be verified that they also satisfy condition (iii).



(a) The token placed at v_1 is $(G_P^{v_1}, \mathbf{I'_1} \cap G_P^{v_1})$-rigid.



(b) The token placed at v_2 is $(G_P^{v_2}, \mathbf{I'_2} \cap G_P^{v_2})$-rigid.

Figure 3: After removing the edges of P

As we understand Lemma 6, it must now follow that P is $(G, \mathbf{I})$-confined, which is a contradiction.

4 Progress on resolving the issue

So far, we have not been able to resolve this issue.

References

[1] Duc A. Hoang and Ryuhei Uehara. Sliding tokens on a cactus. In Seok-Hee Hong, editor, *Proceedings of ISAAC 2016*, volume 64 of *LIPIcs*, pages 37:1–37:26. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016. doi: 10.4230/LIPIcs.ISAAC.2016.37.