Sliding tokens on a cactus

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Outline

- Reconfiguration Problems.
- The SLIDING TOKEN problem for a cactus.
- Interesting open questions.

Reconfiguration Problems

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- INSTANCE:
  1. Collection of configurations.
  2. Allowed transformation rule(s).
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configuration A

8	15	13	3
10		14	7
5	1	2	4
9	12	11	6

configuration B

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 1: The 15-puzzles. (In general, the $(k^2 - 1)$-puzzles.)

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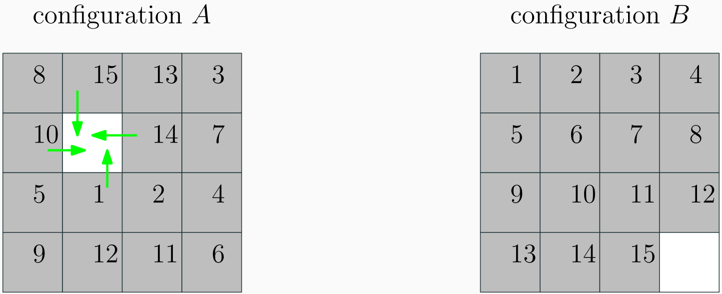

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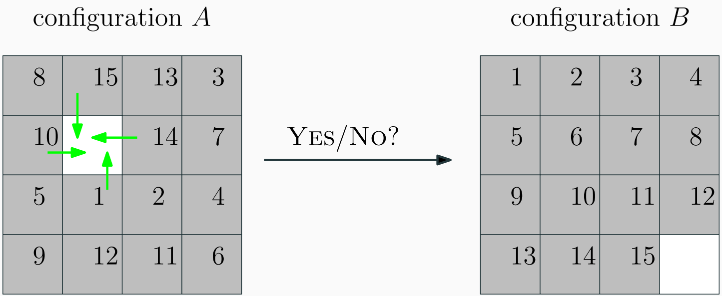

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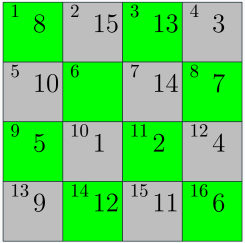

Parity (even/odd) Checking ($O(n)$ time)

$(1, 8, 7, 14, 12, 4, 3, 13, 9, 5, 10)(2, 15, 11)(6, 16)$

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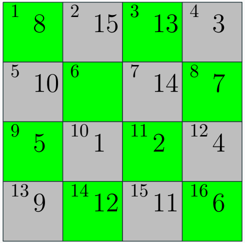

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If YES, need at most $O(n^3)$ moves.

[Kornhauser, Miller, and Spirakis 1984]

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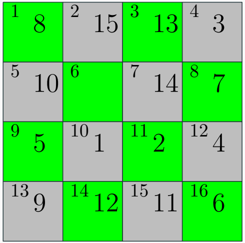

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Find minimum number of moves? - NP-complete

[Ratner and Warmuth 1990]

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Reconfiguration variants have been studied for several well-known problems:

- SATISFIABLILITY,
- INDEPENDENT SET, VERTEX COVER, CLIQUE,
- VERTEX-COLORING, (LIST) EDGE-COLORING,
- and so on.

Recent Survey on Reconfiguration Problems

Jan van den Heuvel (2013). “The complexity of change”. In: *Surveys in Combinatorics 2013*. Ed. by Simon R. Blackburn et al. Cambridge University Press, pp. 127–160

The Sliding Token problem for a cactus

- Instance:
  - Collection of independent sets of a graph.
  - Allowed transformation rule: Token Sliding (TS).
- Question: Decide if there exists a sequence of independent sets (called at TS-sequence) $\mathcal{S} = \langle \mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_\ell \rangle$ that transforms (reconfigures) $\mathbf{I} = \mathbf{I}_1$ to $\mathbf{J} = \mathbf{I}_\ell$, where $\mathbf{I}_{i+1}$ is obtained from $\mathbf{I}_i$ by sliding a token from a vertex $u \in \mathbf{I}_i \setminus \mathbf{I}_{i+1}$ to its neighbor $v \in \mathbf{I}_{i+1} \setminus \mathbf{I}_i$, $i \in \{1, \dots, \ell - 1\}$.

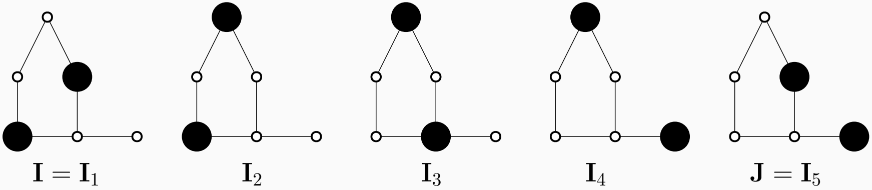

Figure 2: A TS-sequence that reconfigures $\mathbf{I} = \mathbf{I}_1$ to $\mathbf{J} = \mathbf{I}_5$. Vertices of an independent set are marked with black circles (tokens).

Complexity status of Sliding Token

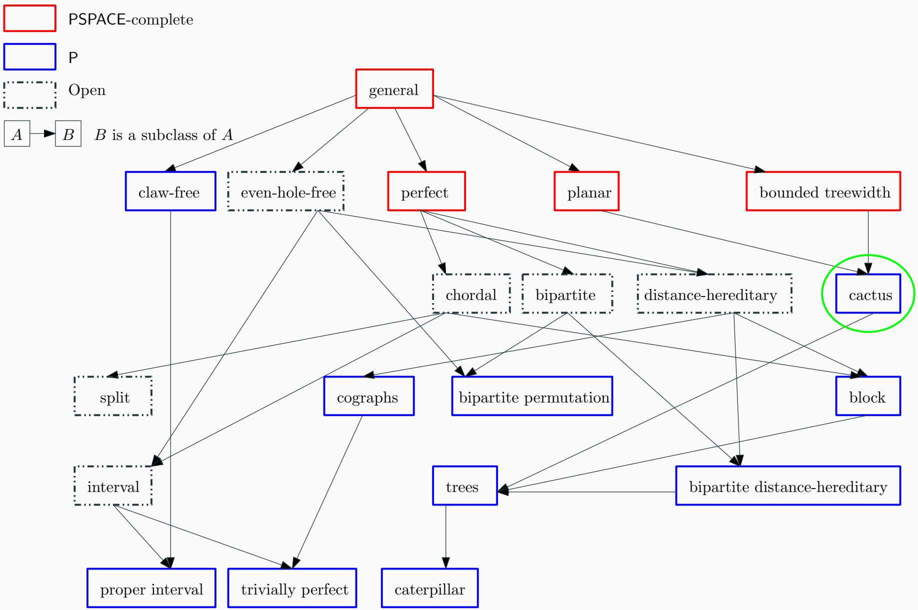

Figure 3: Complexity status of Sliding Token.

A cactus

A cactus is a graph such that every block (i.e., maximal biconnected subgraph) is either an edge or a simple cycle.

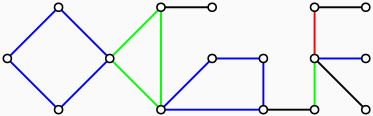

Figure 4: A cactus and its blocks. Two blocks sharing the same vertex are of diffenrent colors.

Why study SLIDING TOKEN for a cactus?

There are a few reasons that motivate our study.

1. We want to understand Intractability vs Polynomial-time tractability of SLIDING TOKEN for bounded-treewidth/planar graphs and their subclasses: Before cacti, the "largest" subclass with polynomial-time tractability is trees.

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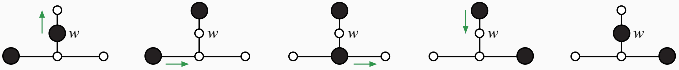
2. Even for trees, a token sometimes needs to make "detours" to preserve the independence property. In general, there might be a YES-instance that requires super-polynomial number of token-slides. (see [Demaine et al. 2015])



(a) $I_b = I_1$ (b) I_2 (c) I_3 (d) I_4 (e) $I_r = I_5$

Figure 5: Detours in a tree.

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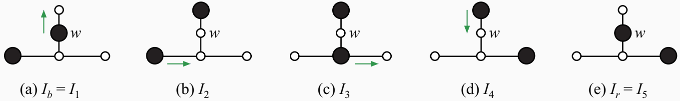


Figure 5: Detours in a tree.

3. In a cactus, there might be more than one path connecting two given vertices. It follows that there might be exponential number of "routes" that a token can be moved.

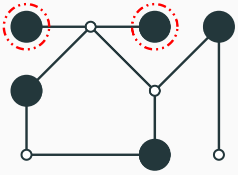
Given an instance $(G, \mathbf{I}, \mathbf{J})$ of SLIDING TOKEN, where $\mathbf{I}$ and $\mathbf{J}$ are independent sets of a cactus G, we can

1. Characterize all structures that forbid the existence of a TS-sequence between $\mathbf{I}$ and $\mathbf{J}$ in polynomial time.
   - A token that cannot be slid at all (called a $(G, \mathbf{I})$-rigid token).
   - A cycle whose inside-tokens form a maximum independent set of it and no token can be slid "out" or "in" (called a $(G, \mathbf{I})$-confined cycle).
2. Prove the existence of a TS-sequence between $\mathbf{I}$ and $\mathbf{J}$ when no such structures exist.

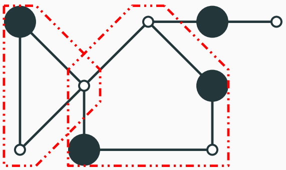
The general idea

Lemma 1

One can find all $(G, \mathbf{I})$-rigid tokens in $O(n^2)$ time, where $n = |V(G)|$. Without $(G, \mathbf{I})$-rigid tokens, one can find all $(G, \mathbf{I})$-confined cycles in $O(n^2)$ time.



(a) $(G, \mathbf{I})$-rigid tokens



(b) $(G, \mathbf{I})$-confined cycles

Figure 6: Examples of the forbidden structures.

Lemma 2

If the set of $(G, \mathbf{I})$-rigid tokens and $(G, \mathbf{J})$-rigid tokens are different, then it is a NO-instance. Without $(G, \mathbf{I})$-rigid and $(G, \mathbf{J})$-rigid tokens, if the set of $(G, \mathbf{I})$-confined cycles and $(G, \mathbf{J})$-confined cycles are different, then it is a NO-instance.

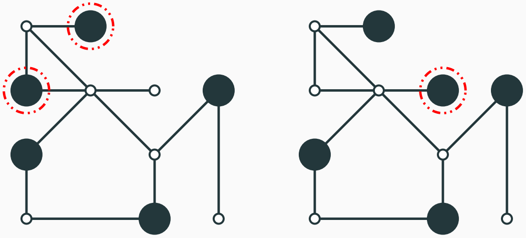


Figure 7: The set of $(G, \mathbf{I})$-rigid tokens and $(G, \mathbf{J})$-rigid tokens are different.

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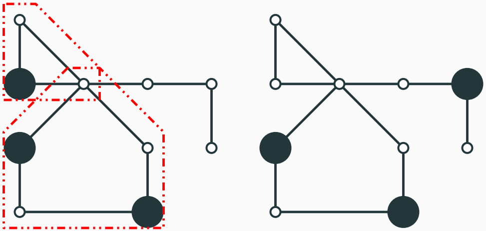

Figure 8: The set of $(G, \mathbf{I})$-confined cycles and $(G, \mathbf{J})$-confined cycles are different.

Lemma 3

Without rigid tokens and confined cycles (for both $\mathbf{I}$ and $\mathbf{J}$), $\mathbf{I}$ can be reconfigured to $\mathbf{J}$ if and only if $|\mathbf{I}| = |\mathbf{J}|$.

Proof Idea: Construct an "intermediate" independent set $\mathbf{I}^*$ such that both $\mathbf{I}$ and $\mathbf{J}$ can be reconfigured to $\mathbf{I}^*$.

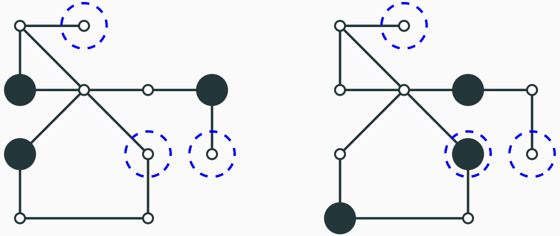

Figure 9: Illustration of **Lemma 3**.

Lemmas 1–3 give rise to the following polynomial-time algorithm. For an instance $(G, \mathbf{I}, \mathbf{J})$ of SLIDING TOKEN, where G is a cactus and $\mathbf{I}$, $\mathbf{J}$ are two independent sets of G.

- **Step 1:**
  - **Step 1-1:** If the set of $(G, \mathbf{I})$-rigid tokens and $(G, \mathbf{J})$-rigid tokens are different, return NO.
  - **Step 1-2:** Otherwise, remove all vertices where $(G, \mathbf{I})$-rigid tokens are placed and its neighbors, and go to **Step 2**. Let G' be the resulting graph.
- **Step 2:**
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Interesting open questions

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1. SLIDING TOKEN for bipartite graphs is still open. Unlike a cactus, two cycles of a bipartite graph may have more than one vertex in common. Polynomial results are known for bipartite permutation graphs [Fox-Epstein et al. 2015].
2. Given a YES-instance, finding a shortest TS-sequence is open even for trees. The only known polynomial result regarding this problem is the case for caterpillars [Yamada and Uehara 2016].
3. It is interesting to find a graph class $\mathcal{G}$ with the property that SLIDING TOKEN is polynomial-time solvable for $\mathcal{G}$, and finding a shortest TS-sequence for $\mathcal{G}$ is NP-hard. We conjecture that $\mathcal{G}$ might be cacti.

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