On Token Sliding (Reconfiguration) Graphs of Independent Sets

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Based on joint works with David Avis (Kyoto University, Japan)

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Outline

A Geometric Example

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- *Flip graphs* are very important in computational geometry and have many applications in various settings, including
  - measuring the "similarity" (or "distance") between two geometric objects by constraining the family of "allowable flips";
  - enumerating geometric objects;
  - computing geometric objects with respect to some optimization criteria;
  - and so on

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  - and so on
- For a given graph G and a fixed integer $k \geq 1$, the TS_k*-(reconfiguration) graph* (or TS_k*-graph*) of G, denoted by $\mathsf{TS}_k(G)$, is the graph whose
  - *vertices* are *independent sets* (or *stable sets*, which are vertex subsets of pairwise non-adjacent vertices) of G
    - Each stable set can be seen as a *set of tokens placed on vertices of G*
  - *edges* are defined by *Token Sliding* (TS)

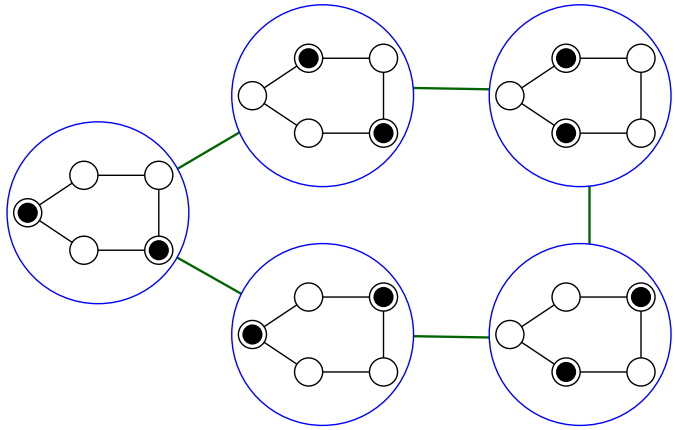

Figure: $\mathsf{TS}_2(C_5)$

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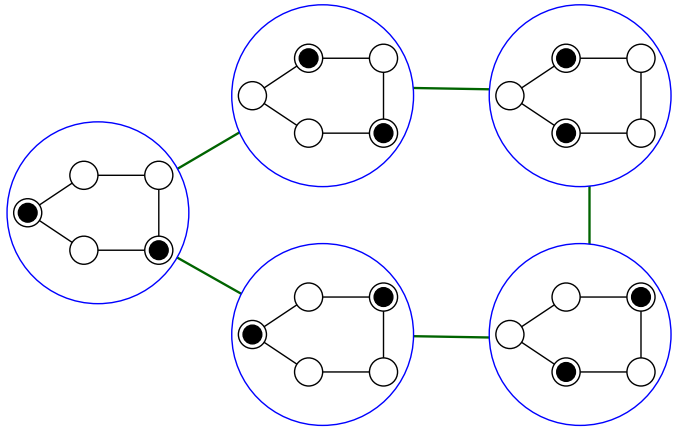

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- We present an example showing *a relationship between some flip graphs of triangulations and* TS_k*-graphs (of some graphs)*

A Geometric Example

- Given a set P of n points in the plane, no three collinear and no four co-circular

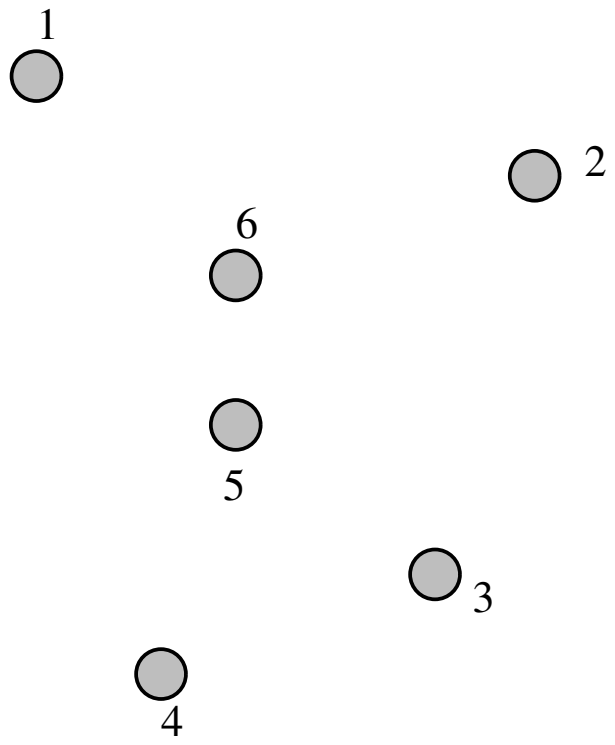

Figure: A set P of n points in the plane.

A Geometric Example

- Given a set P of n points in the plane, no three collinear and no four co-circular
- Two line segments are *intersecting* if they cross each other at an interior point of each segment, and *non-intersecting* otherwise

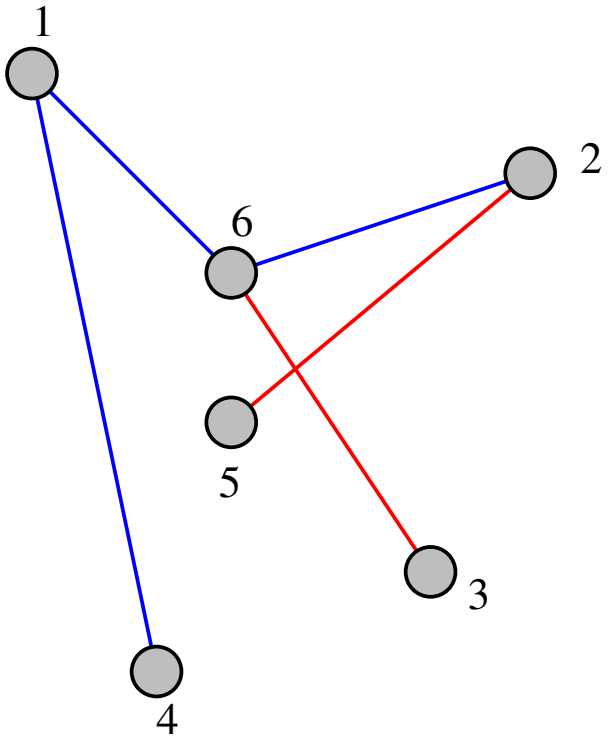

Figure: A set P of n points in the plane. Red (Blue) segments are pairwise *intersecting* (*non-intersecting*).

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- Given a set P of n points in the plane, no three collinear and no four co-circular
- Two line segments are *intersecting* if they cross each other at an interior point of each segment, and *non-intersecting* otherwise
- A *triangulation* of P is any maximal set of pairwise non-intersecting segments. It is well-known that *all triangulations have the same number of edges (segments)*

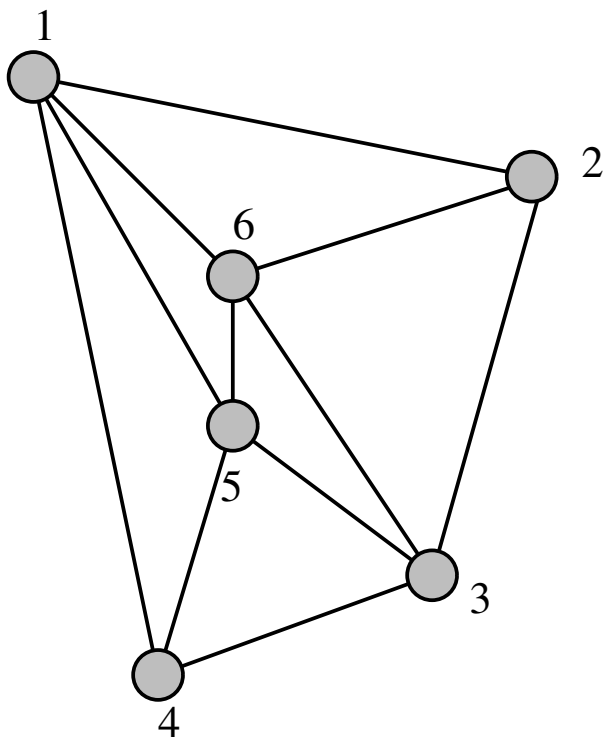

Figure: A set P of n points in the plane. A triangulation of P where each segment is colored in black

A Geometric Example

- The *edge-intersection graph G of P* is the graph whose
  - *vertices* are *the line segments with endpoints in P that intersect at least one other segment*

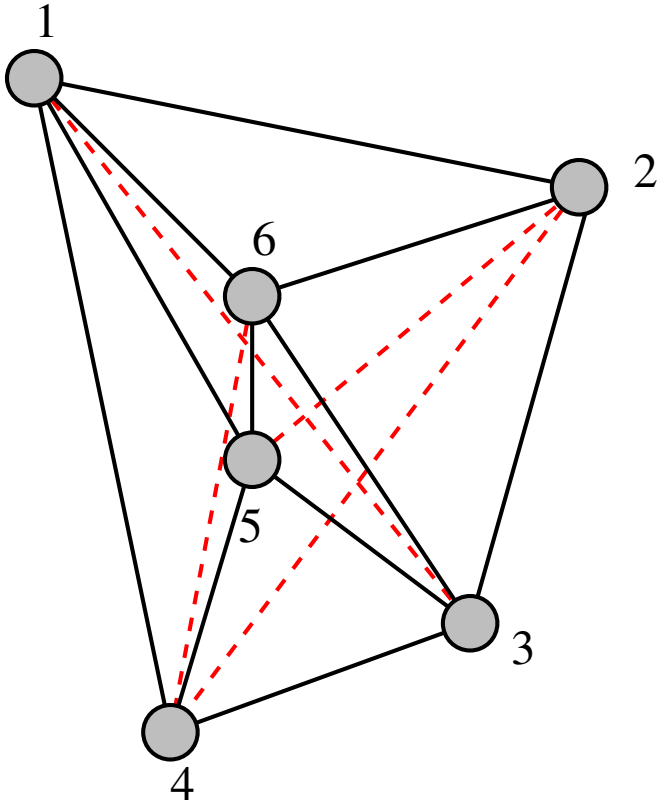

Figure: A triangulation of P is marked by black edges. The remaining line segments are marked by dashed red edges

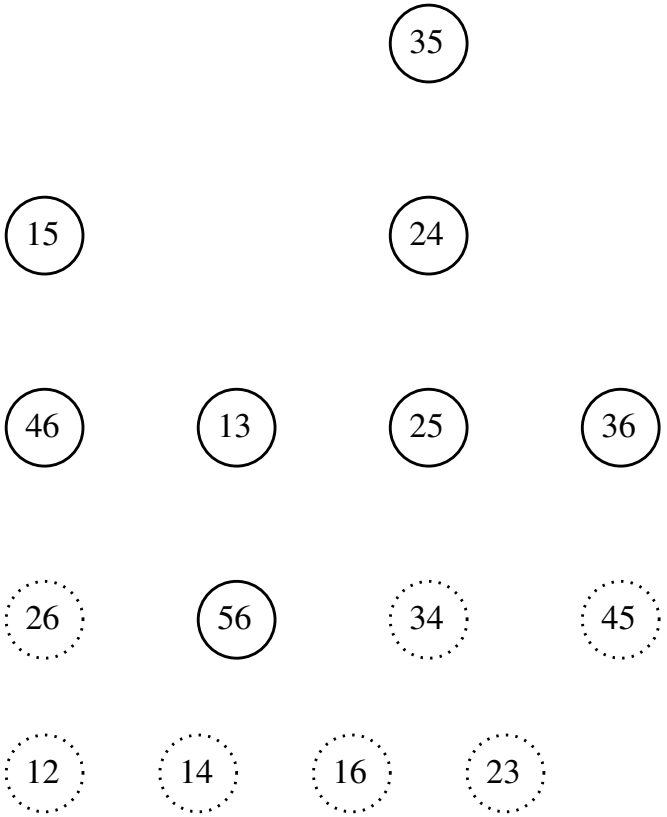

Figure: The edge-intersection graph G of P. Vertices (segments) not in G are depicted by dotted circles.

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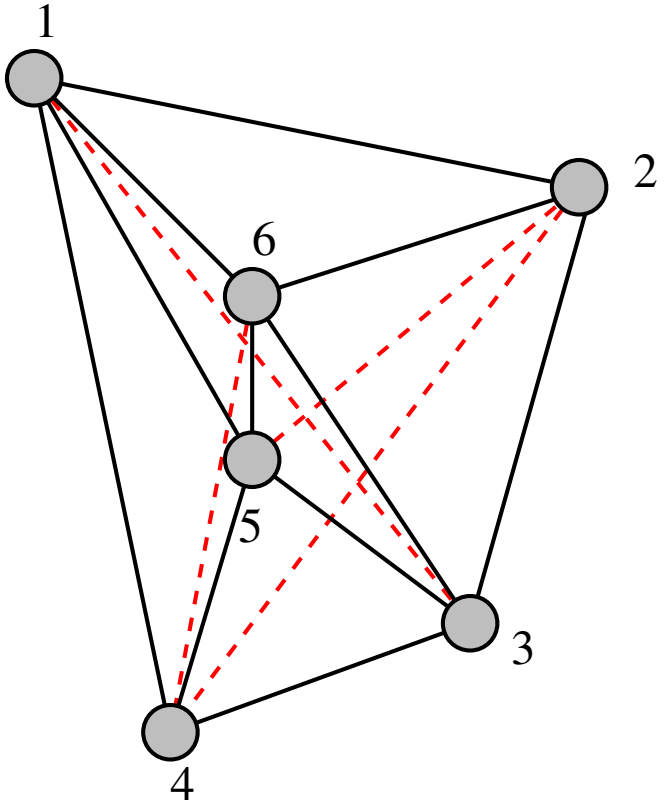
  - *edges* are defined between two vertices whose *corresponding line segments are intersecting*



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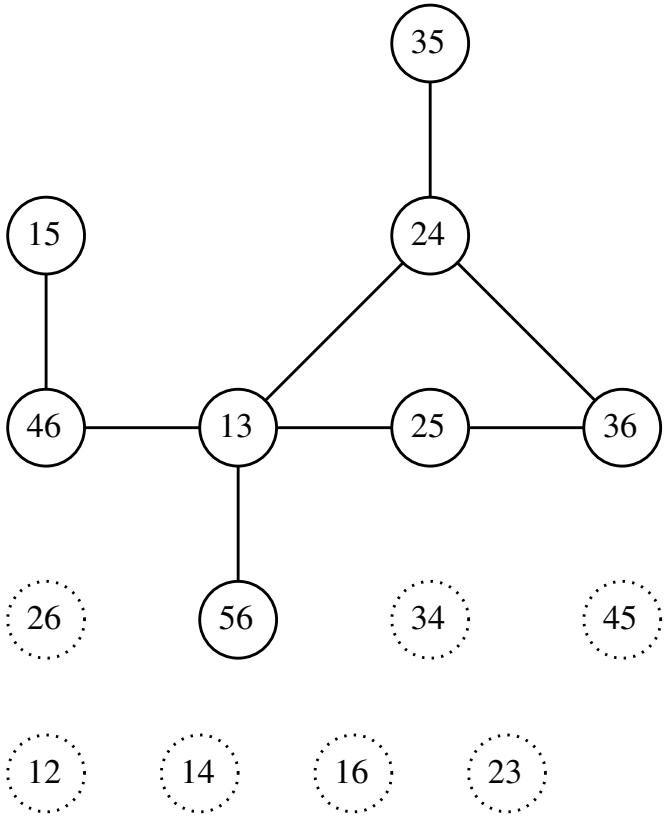

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A Geometric Example

- The *flip graph of triangulations of P* is the graph whose
  - *vertices* are *triangulations of P*
  - *edges* are defined via the *flip operation*: two triangulations T_1, T_2 of P are adjacent if one can be obtained from the other by *flipping* the diagonal of a convex quadrilateral

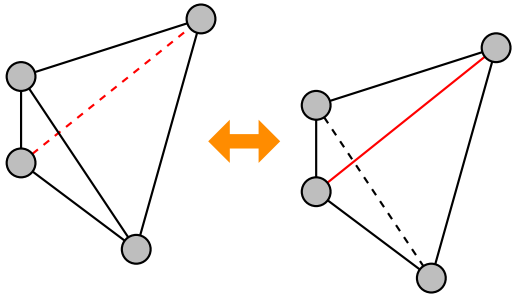


Figure: The flip operation

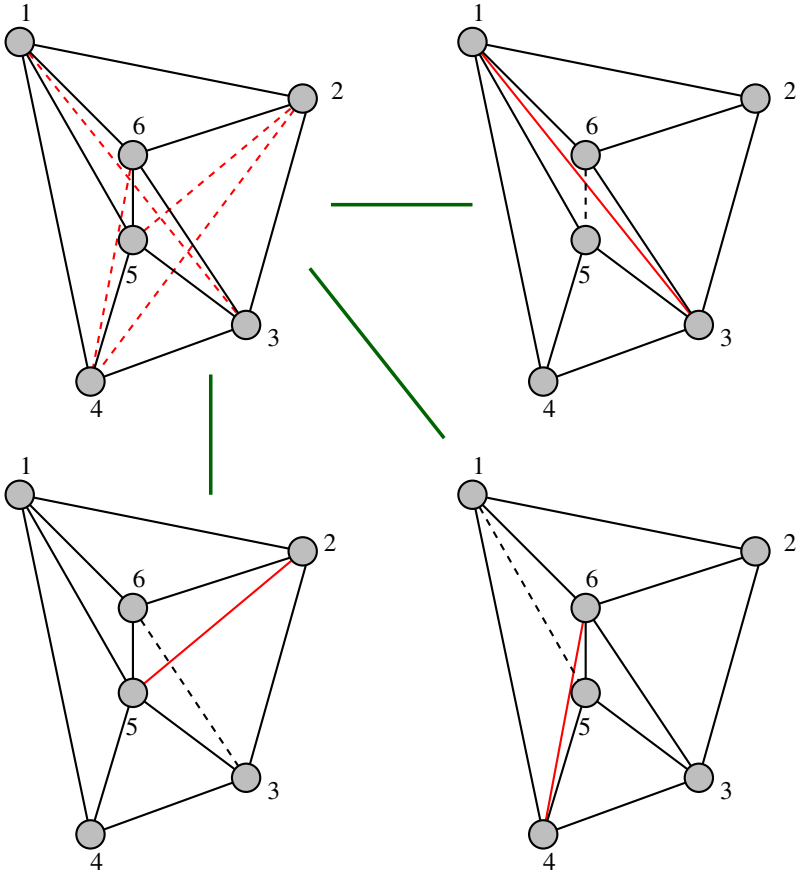
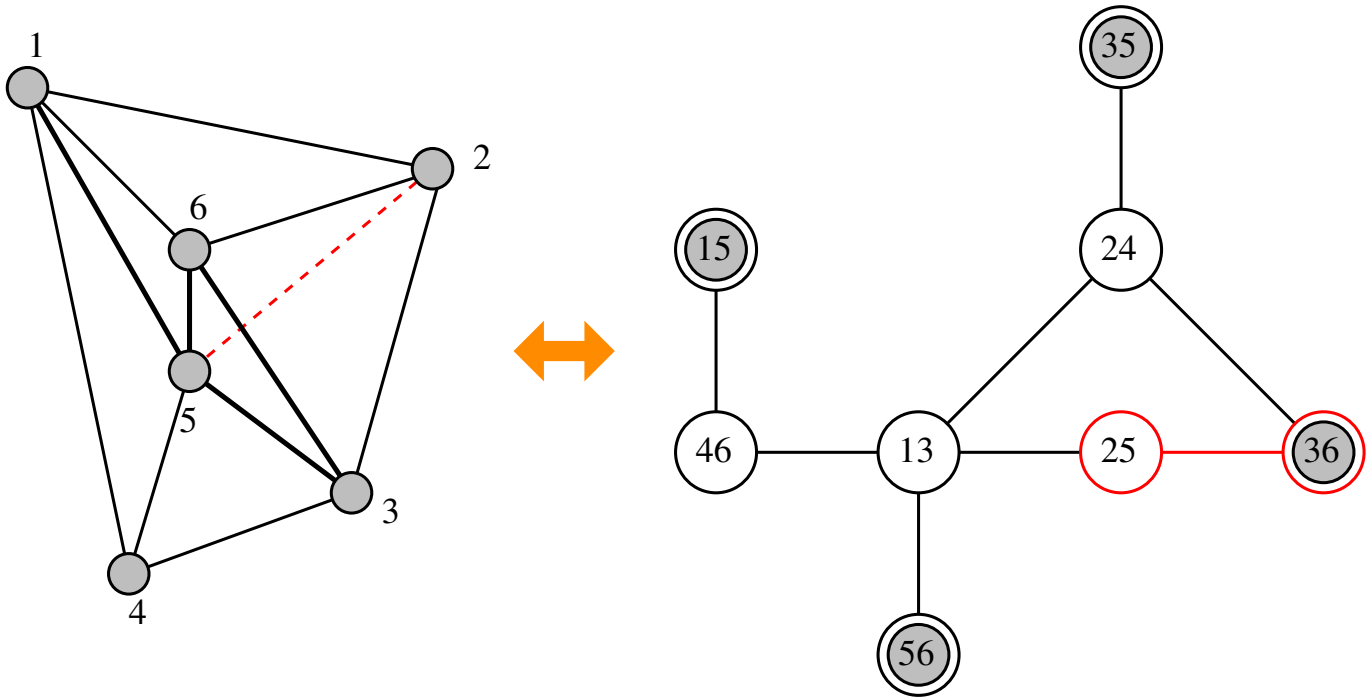


Figure: A vertex (triangulation) in the flip graph of triangulations of P (top left) and its adjacent ones

A Geometric Example

- A set of k pairwise non-intersecting segments where each member intersects at least one other segment $\Leftrightarrow$ A vertex of $\mathsf{TS}_k(G)$ (e.g., size-k stable set of G)
- An edge of the flip graph of triangulations of P $\Leftrightarrow$ An edge of $\mathsf{TS}_k(G)$

Flip graph of triangulations of P $\Leftrightarrow$ $\mathsf{TS}_{\alpha(G)}$-graph of the edge-intersection graph G of P



Note: $\alpha(G)$ denotes the size of a maximum stable set of G

TS_k-Graphs: Background and Motivation

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- Related works involve *Token Graphs*
  - A *k-token graph* $F_k(G)$ of a graph G is the graph whose
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  - Under the "reconfiguration setting", $\mathsf{TS}_k(G)$ has been *extremely well-studied* from the *algorithmic perspective* [van den Heuvel 2013]; [Nishimura 2018]; [Bousquet, Mouawad, Nishimura, and Siebertz 2022]

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- We *initiate* the study of $\mathsf{TS}_k(G)$ from a purely *graph-theoretic perspective* [Avis and Hoang 2023a]; [Avis and Hoang 2023b]

Paths Are TS_k-Graphs

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- We say that a graph H is a TS_k*-graph* if there exists a graph G such that $H \simeq \mathsf{TS}_k(G)$
  - For example, C_5 is a TS_2-graph, since $C_5 \simeq \mathsf{TS}_2(C_5)$
- Since $G \simeq \mathsf{TS}_1(G)$ for any graph G, *the case $k = 1$ is trivial and uninteresting*. From now on, we will always consider *fixed integers $k \geq 2$*

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Theorem 1

A path P_n $(n \geq 1)$ is a TS_k-graph for any $k \geq 2$

- We briefly explain *three different ways* of proving Theorem 1, each of which *indicates a way of looking at TS_k-graphs*

Line Graphs and TS_2-Graphs

Lemma 2

Let $\overline{G}$ and $L(G)$ be respectively the complement and line graph of G

(a) $\mathsf{TS}_2(\overline{G})$ *is a (spanning) subgraph of* $L(G)$

(b) $\mathsf{TS}_2(\overline{G}) \simeq L(G)$ *if and only if G is triangle-free*

edges of G ⟷ non-edges of $\overline{G}$

⇕ ⇕

vertices of $L(G)$ ⟷ vertices of $\mathsf{TS}_2(\overline{G})$

edges of a triangle of G ⟷ three size-2 stable sets of $\overline{G}$

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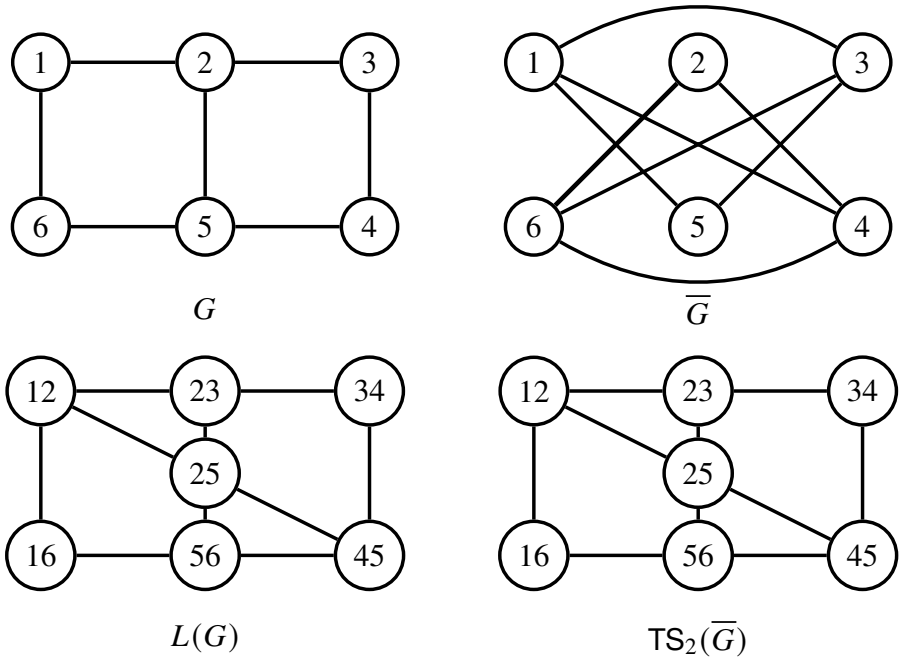
vertices of a triangle of $L(G)$ ⟷ a size-3 stable set of $\mathsf{TS}_2(\overline{G})$



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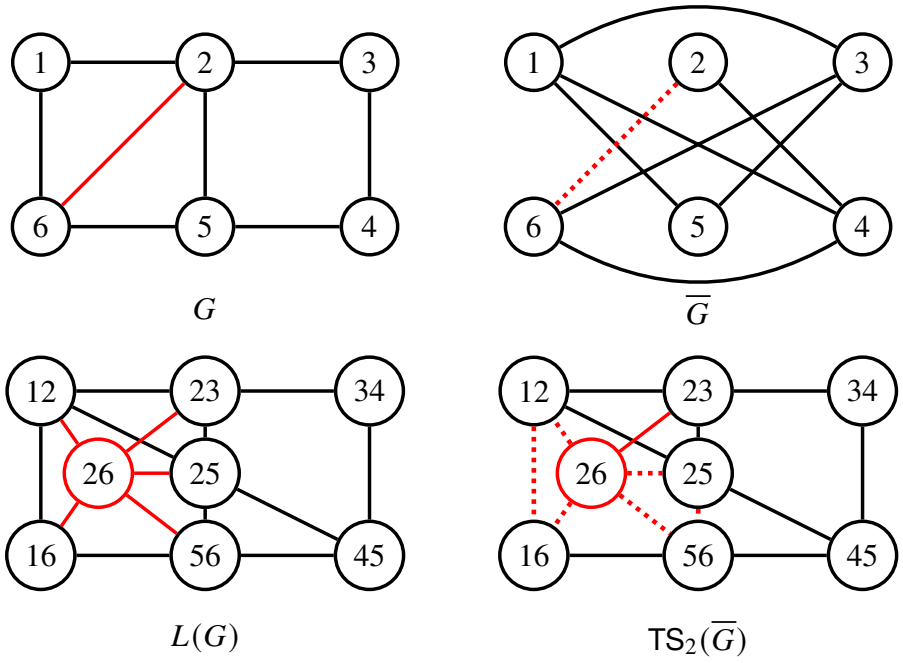

Line Graphs and TS_2-Graphs

- Since P_n *is triangle-free* for any $n > 1$ and $P_n \simeq L(P_{n+1})$, Lemma 2 immediately gives us

$$P_n \simeq L(P_{n+1}) \simeq \mathsf{TS}_2(\overline{P_{n+1}})$$

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- The following proposition *settles the case $k \geq 3$* (note that $\alpha(\overline{P_{n+1}}) = 2$)

Proposition 3

Given H and let $G = \mathsf{TS}_{\alpha(H)}(H)$. Then, for every $k \geq \alpha(H)$, G is a TS_k-graph

- Construct a graph H' by taking the *disjoint union of H and exactly $k - \alpha(H)$ isolated vertices*
- Then, $G = \mathsf{TS}_{\alpha(H)}(H) \simeq \mathsf{TS}_k(H')$ for any fixed integer $k \geq \alpha(H)$

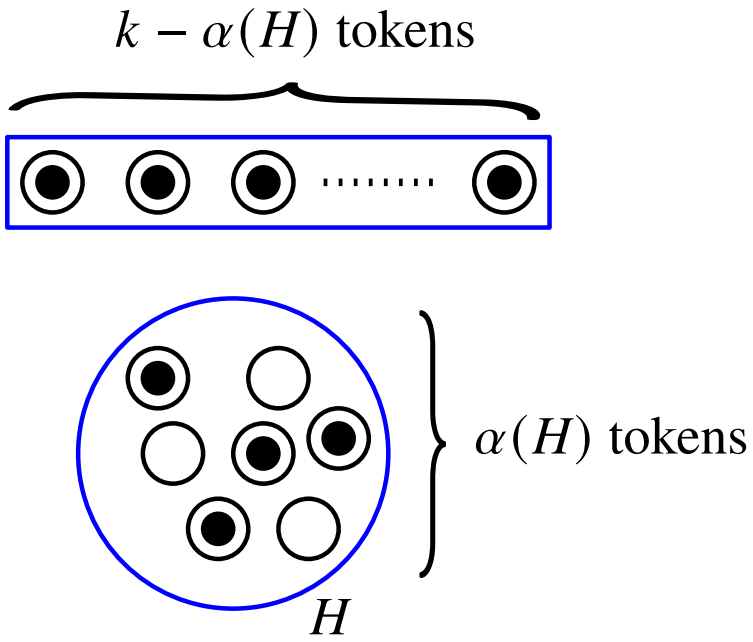
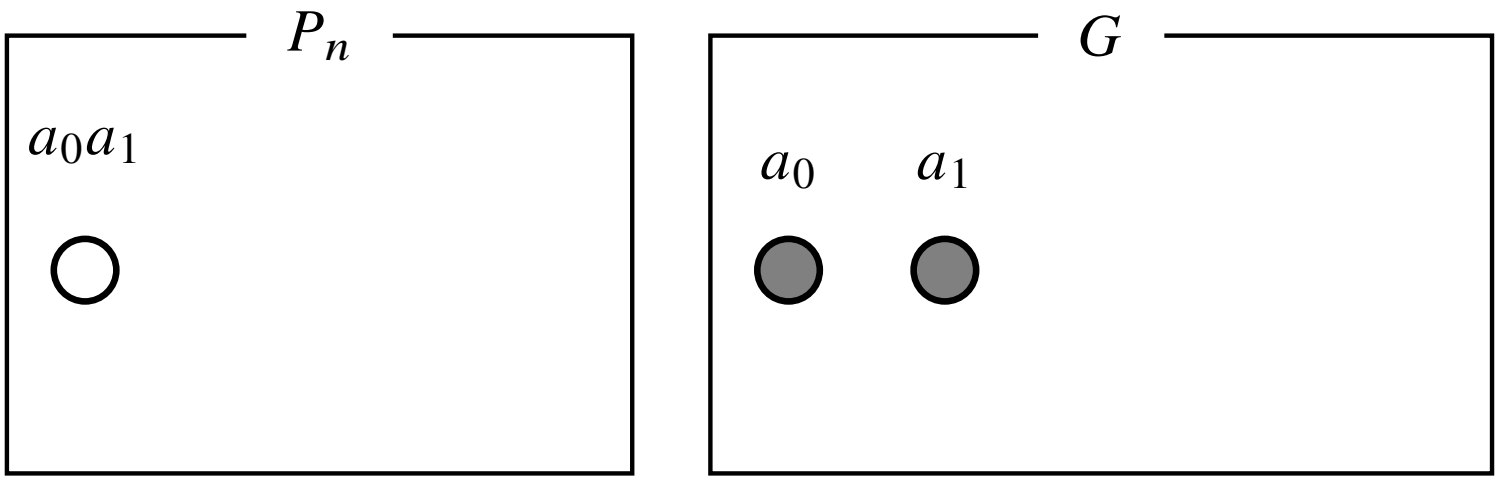


Figure: The graph H'

Labelling Vertices By Stable Sets

Key Idea ($k = 2$)

There exists G such that $P_1 \simeq \mathsf{TS}_2(G)$. For $n \geq 2$, we *update G and label vertices of P_n by size-2 stable sets of G*



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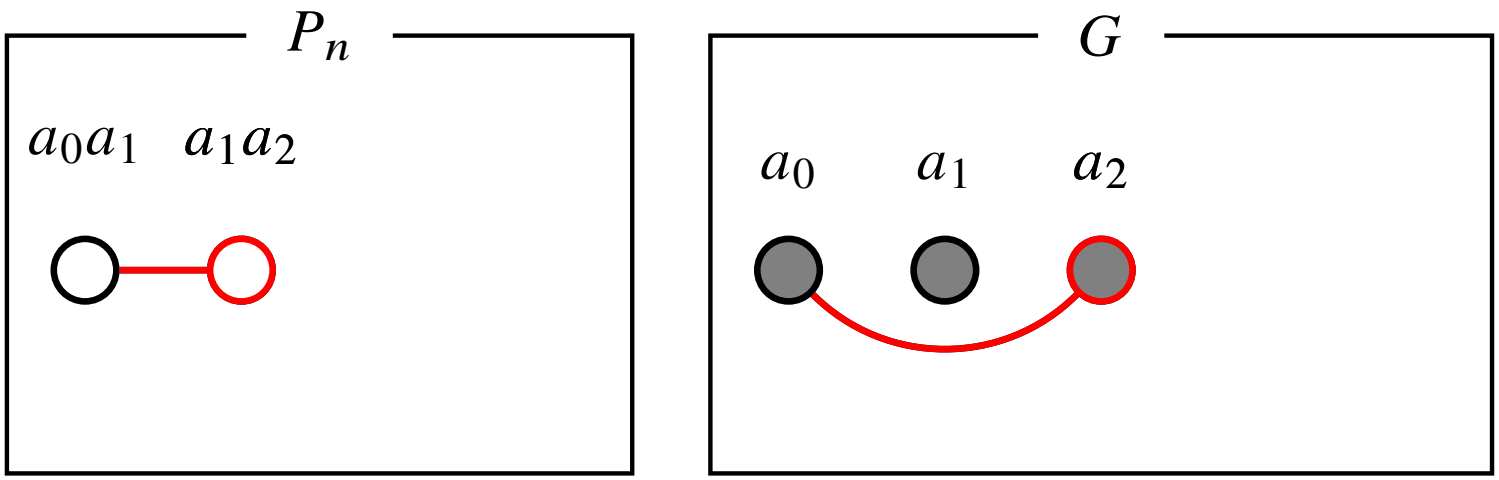

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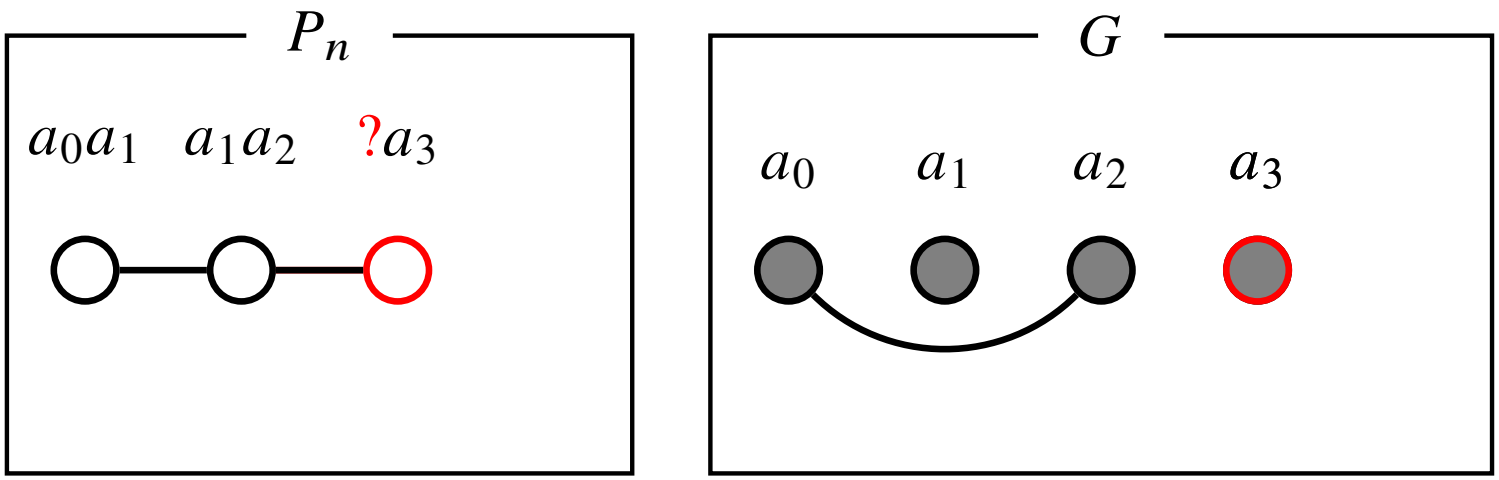

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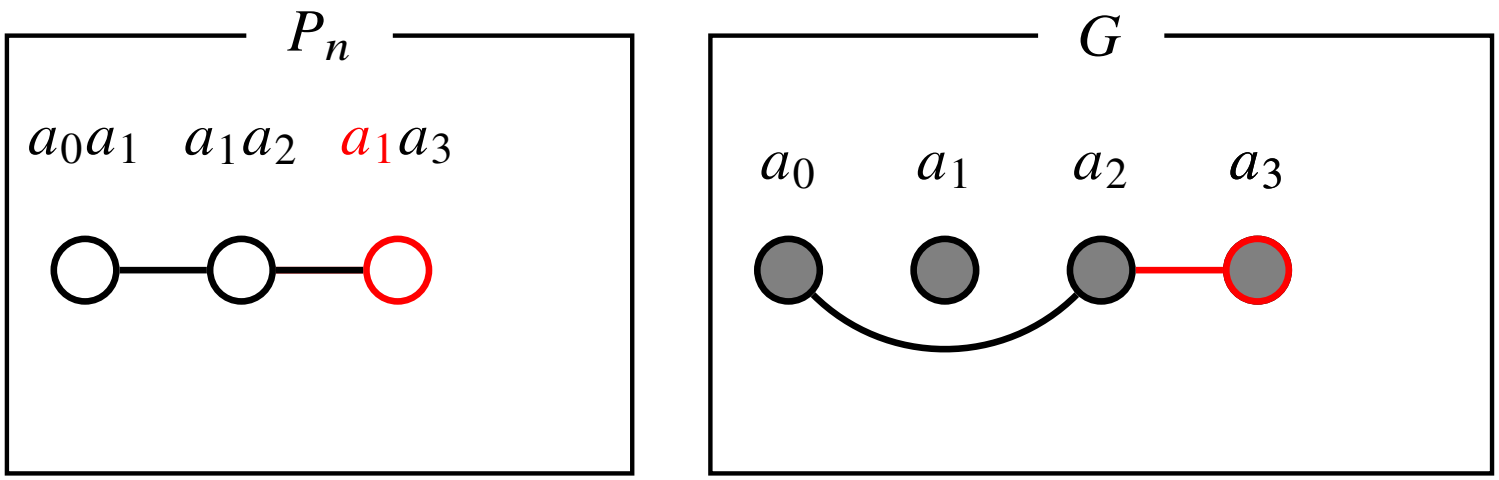

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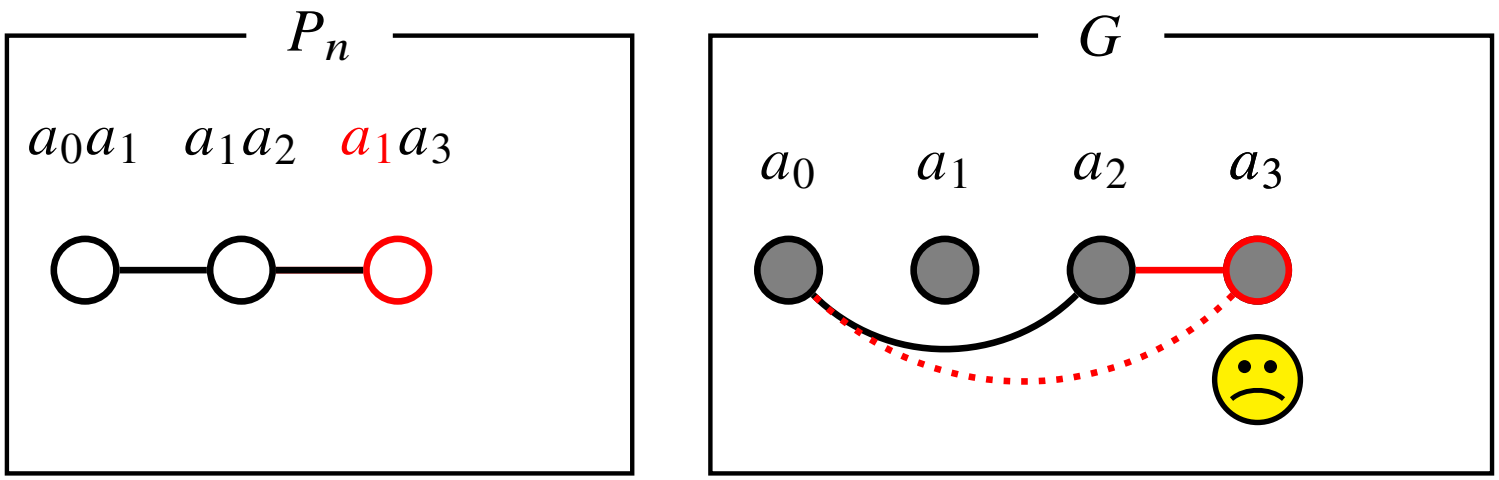

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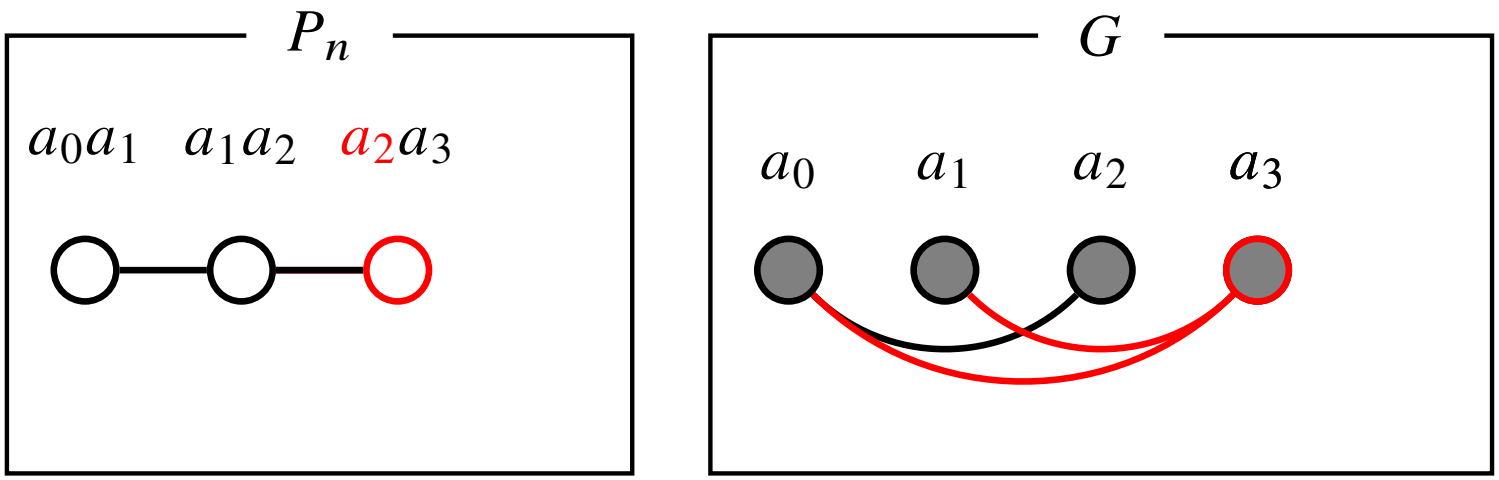

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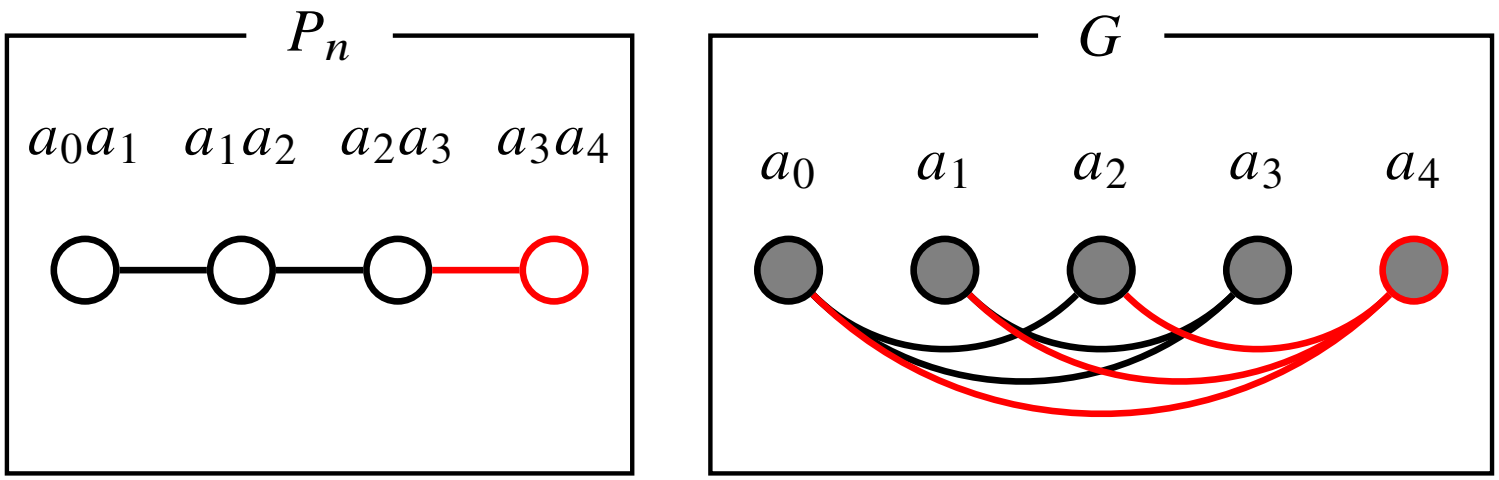


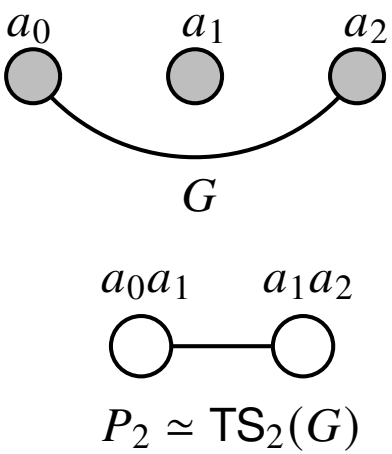
Figure: $P_n \simeq \mathsf{TS}_2(G)$

Gluing Labelled Graphs Together

Key Idea ($k = 2$)

(a) There exists a (labelled) graph G such that $P_2 \simeq \mathsf{TS}_2(G)$

(b) With a "good" vertex-labelling (by stable sets of some graph), the TS_2-graph P_n can be constructed by *"gluing" two smaller* TS_2*-graphs* P_{n-1} *and* P_2, for any $n \geq 3$



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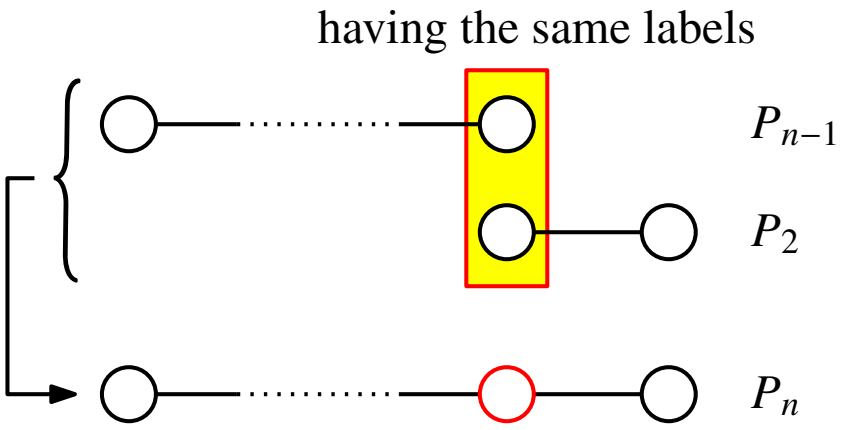
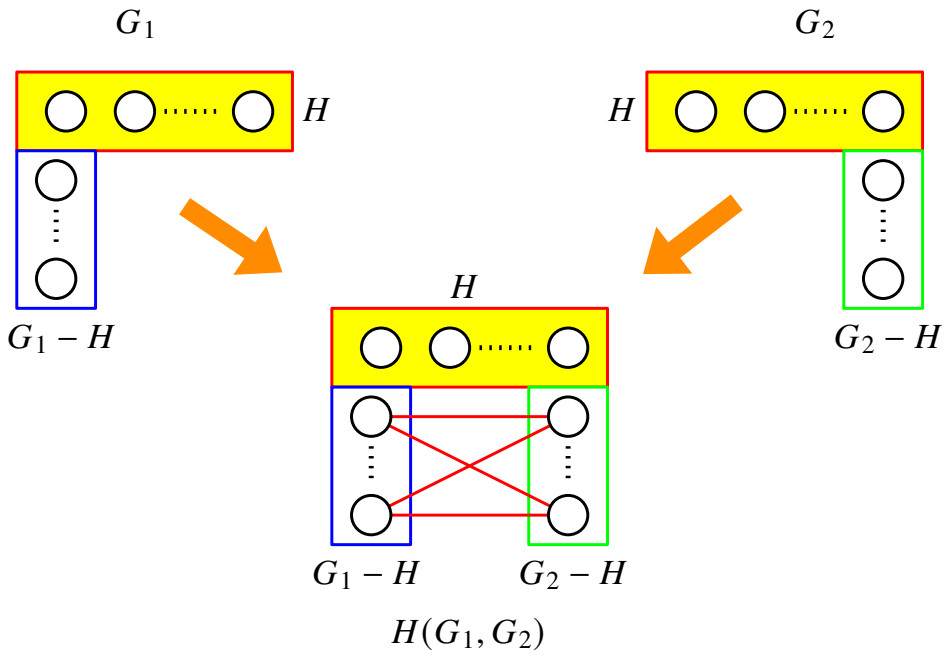


(b)

Gluing Labelled Graphs Together

- Vertex-labelled graphs G_1 and G_2 are *H-consistent* if *the (possibly empty) intersection of their vertex sets defines the same (possibly empty) common induced subgraph H*
- The *H-join of H-consistent graphs G_1, G_2*, denoted by $H(G_1, G_2)$, is the graph constructed by
  - Putting G_2 on top of G_1 (or vice versa) so that their common induced subgraph H coincide
  - Joining any pair of vertices v, w between $G_1 - H$ and $G_2 - H$, respectively



Gluing Labelled Graphs Together

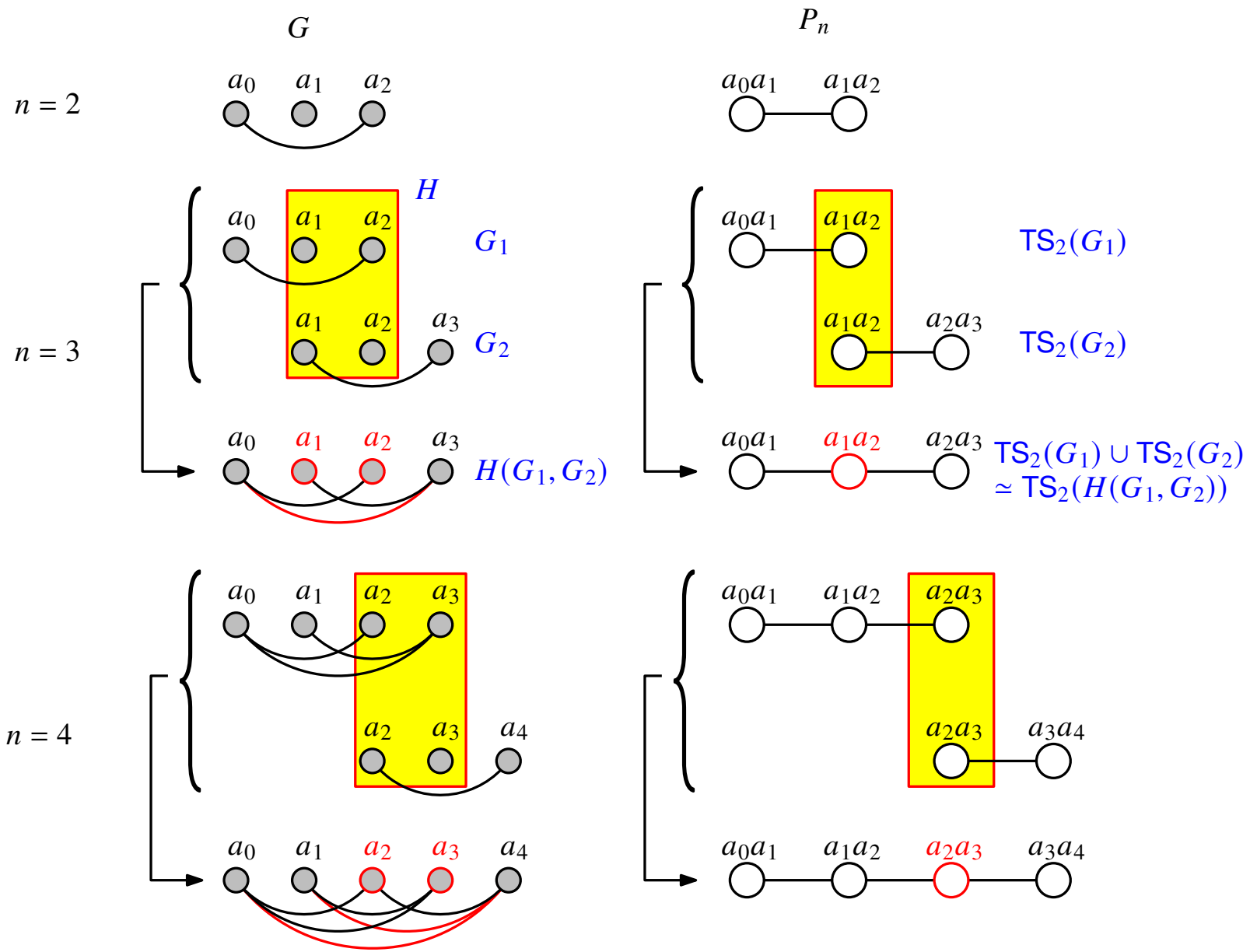


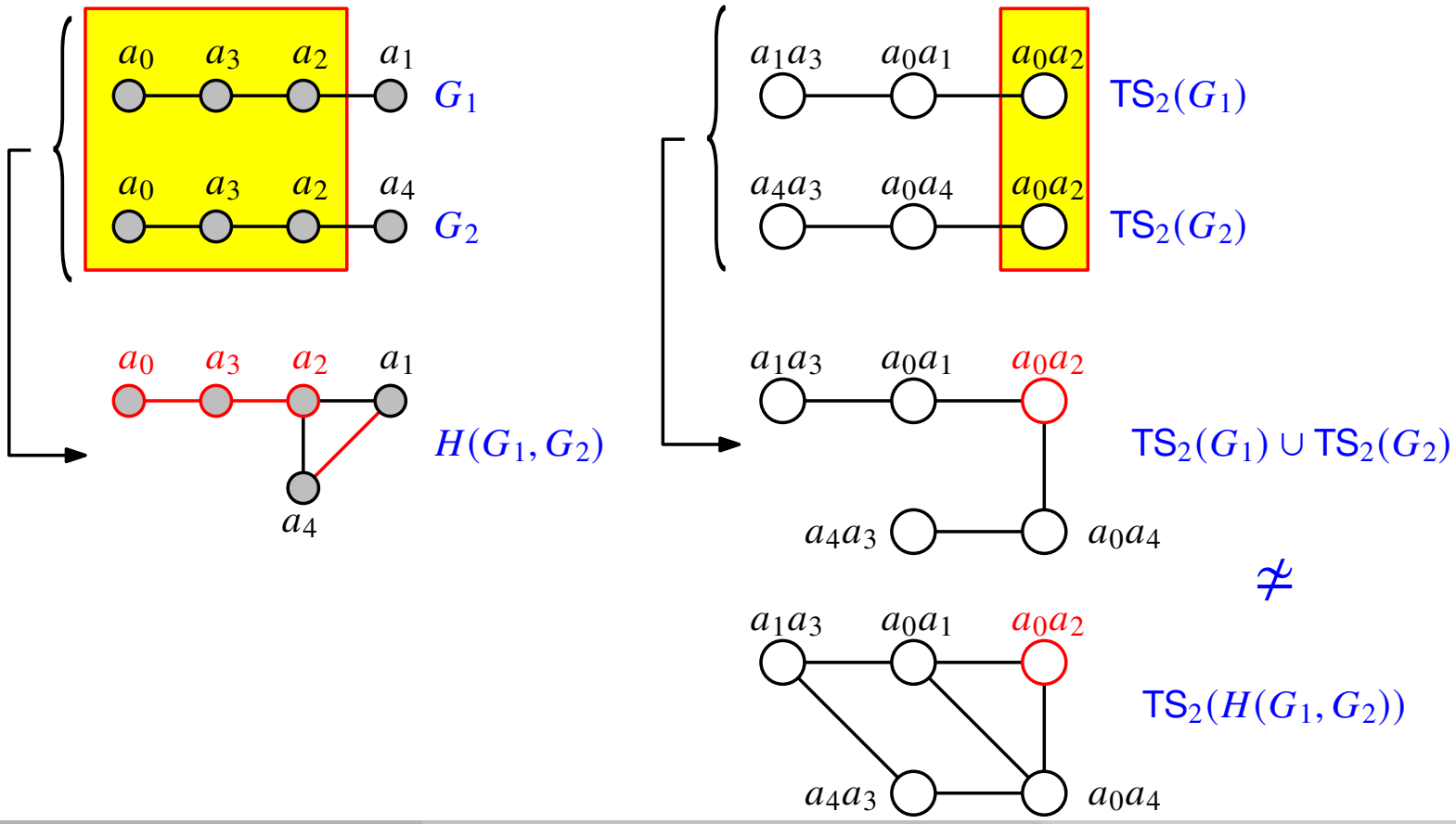
Figure: $P_n \simeq \mathsf{TS}_2(G)$

Gluing Labelled Graphs Together

Remark

There exist H-consistent graphs G_1, G_2 such that

$$\mathsf{TS}_2(G_1) \cup \mathsf{TS}_2(G_2) \not\approx \mathsf{TS}_2(H(G_1, G_2))$$



Gluing Labelled Graphs Together

In general, the following proposition describes *how to compute the* TS_k*-graph of an* H*-join*

Proposition 4

Let $k \geq 2$ *and let* G_1 *and* G_2 *be two* H*-consistent graphs.* $\mathsf{TS}_k(H(G_1, G_2))$ *is the union of* $\mathsf{TS}_k(G_1)$, $\mathsf{TS}_k(G_2)$ *and for every pair of* k*-element stable sets* S_1 *in* G_1 *and* S_2 *in* G_2 *satisfying*

$$|S_1 \cap V(H)| = |S_2 \cap V(H)| = |S_1 \cap S_2| = k - 1, \tag{1}$$

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- We say that $H(G_1, G_2)$ is *k-crossing free* if there are *no k-element stable sets satisfying condition (1) of Proposition 4*

Corollary 5

Let $k \geq 2$ *and let* G_1 *and* G_2 *be two* H*-consistent graphs.* $H(G_1, G_2)$ *is* k*-crossing free if and only if*

$$\mathsf{TS}_k(H(G_1, G_2)) \simeq \mathsf{TS}_k(G_1) \cup \mathsf{TS}_k(G_2) \tag{2}$$

On TS_k-Graphs

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- If H is an induced subgraph of G then $\mathsf{TS}_k(H)$ is also an induced subgraph of $\mathsf{TS}_k(G)$. The reverse does not hold (e.g., take $H = C_{2k}$ and $G = K_{1,k+1}$) for any $k \geq 2$

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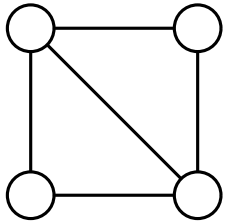
- For any $k \geq 2$,
  - *K_1 is the smallest graph that is a TS_k-graph*
  - On the other hand, the *diamond* is the *smallest graph that is not a TS_k-graph*

G	H such that $G \simeq \mathsf{TS}_k(H)$ $(k \geq 2)$
 K_1	 k isolated vertices
 diamond	does not exist

On Acyclic TS_k-Graphs

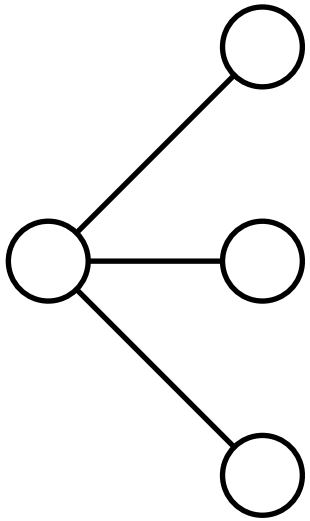
On Acyclic TS_k-Graphs

Remark

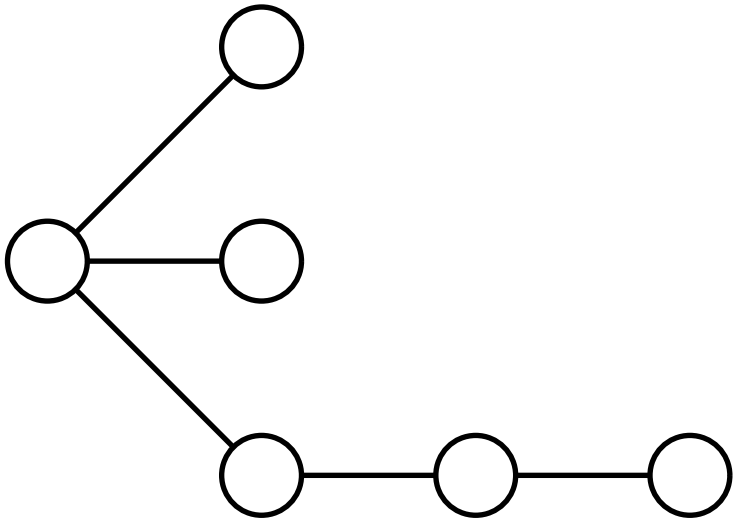
"Being a TS_k-graph" is *not* hereditary, even when $k = 2$

Using the "labelling vertices by stable sets" approach, we showed that

- $K_{1,3}$ is *not* a TS_2-graph. More generally, *$K_{1,n}$ is a TS_k-graph for some fixed $k \geq 2$ if and only if $n \leq k$*
- Replacing an edge of $K_{1,3}$ by a P_4 results a TS_2-graph



not a TS_2-graph



a TS_2-graph

On Acyclic TS_k-Graphs

Open Question

(1) Under which conditions a given graph G satisfies $\mathsf{TS}_k(G)$ is acyclic?

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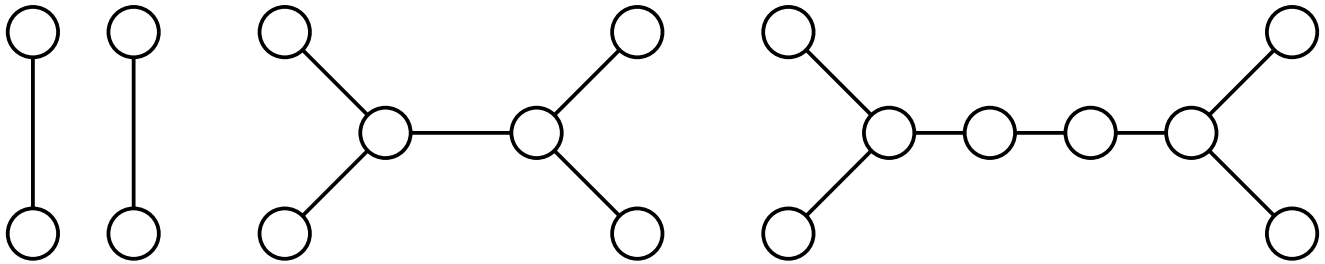

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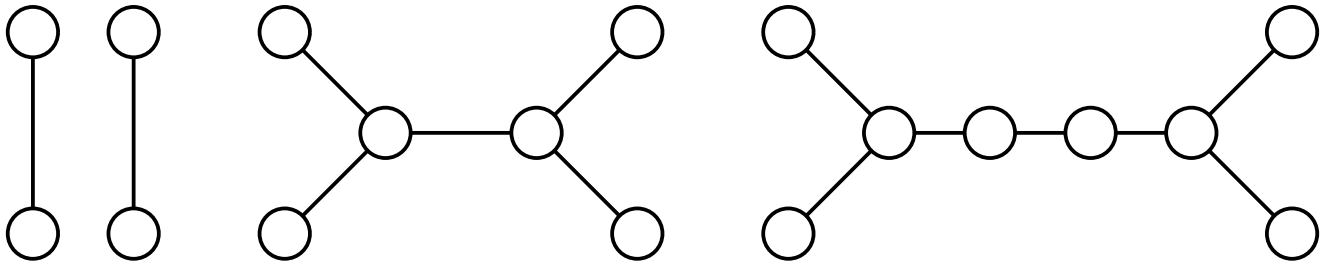

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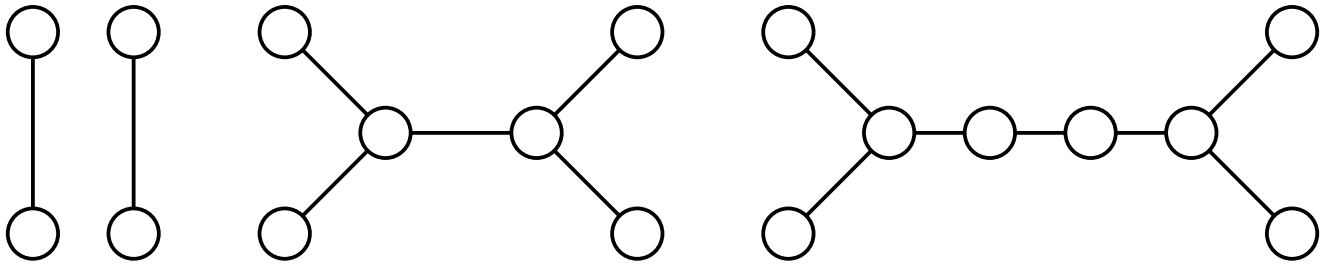
  - **Conjecture:** $\mathsf{TS}_k(F)$ is acyclic if and only if F is $\{2K_2 + (k-2)K_1, D_{2,2,2} + (k-2)K_1, D_{2,4,2} + (k-3)K_1\}$-free, for $k \geq 4$



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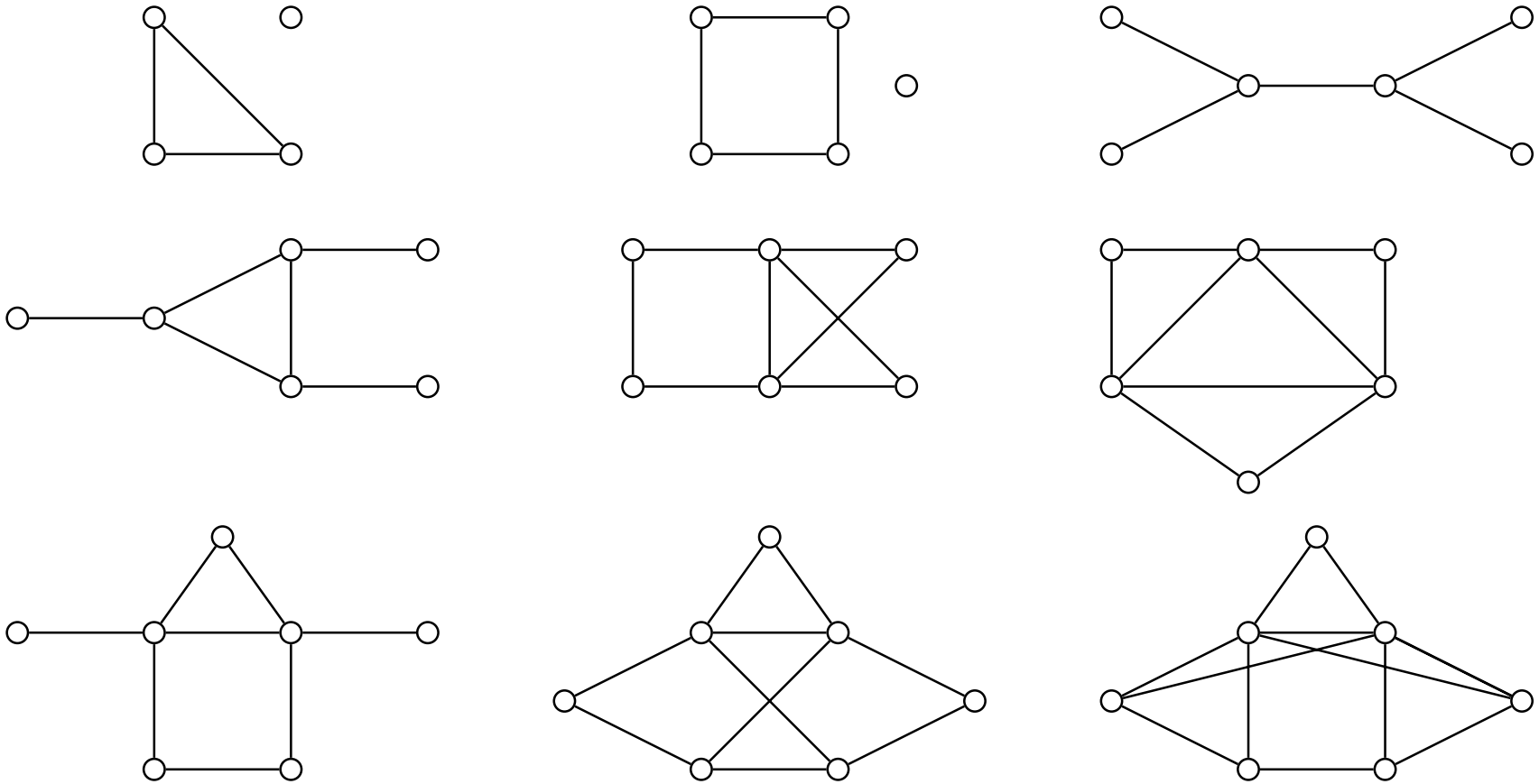
On Acyclic TS_k-Graphs

- Given a graph G. If G contains either $\overline{C_n}$ *($n \geq 4$)* or *one of the following nine graphs* as an induced subgraph then $\mathsf{TS}_2(G)$ has a cycle
  - The $\overline{C_n}$ ($n \geq 4$) graphs come from the "line graph" approach (Lemma 2)
  - (Most of) The below nine graphs come from a computer program



- **Open Question:** Does the reverse hold? (i.e., Did we miss any graph?)

On Acyclic TS_k-Graphs

Using the "gluing graphs together" approach, we showed that

- A *n-ary tree* is a rooted tree in which each node has at most n children. (A 2-ary tree is the well-known binary tree.) *For $k \geq 2$, any k-ary tree is a TS_{k+1}-graph*

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- There exists a graph G such that $\mathsf{TS}_k(G)$ *is a tree containing $K_{1,n}$* if $n \leq 2k$
- **Open Question:** Does there exist a graph G such that $\mathsf{TS}_k(G)$ contains $K_{1,n}$ for $n > 2k$?

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  - However, *the reverse does hold for $n = 2$*, that is, $D_{r,2,s}$ is a TS_k-graph if and only if $s \leq k - 1$

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