Sliding tokens on unicyclic graphs

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概 要

Given two independent sets **I** and **J** (having the same cardinality) of a graph G, and imagine that a token (coin) is placed on each vertex in **I**. Then, the SLIDING TOKEN problem asks if one could transforms **I** to **J** using a sequence of elementary steps, where each step requires sliding a token from one vertex to one of its neighbors such that the resulting set of vertices where tokens are placed still remains independent. Interestingly, on some graph classes such as trees, bipartite graphs, unicyclic graphs, etc., sometimes the tokens are required to make "detours" in order not to violate the independence property. This makes the SLIDING TOKEN problem more complicated and challenged. In this paper, based on the idea of Demaine et al. [3], we present a polynomial-time algorithm for solving SLIDING TOKEN on unicyclic graphs.
Keywords: reconfiguration, independent set, unicyclic graphs, token sliding.

1 Introduction

Reconfiguration problems are the set of problems in which we are given a set of feasible solutions of a problem, together with some reconfiguration rule(s). The question is, using a reconfiguration rule, can we find a step-by-step transformation which transform one solution to another? A well-known example is the SATISFIABILITY reconfiguration problem.More precisely, given two specified satisfiable assignments (assignments which return the TRUE value) A and B to a Boolean formula, one might ask whether A can be transformed into B by changing the assignment of one variable at a time such that each intermediate assignment is also satisfiable. A brief introduction to this reconfiguration framework can be found in [7].

Recently, among many variants of reconfiguration problems, the reconfigurability of INDEPENDENT SET and its related problems, such as the reconfigurability of CLIQUE VERTEX COVER , etc., have been studied extensively. Recall that an *independent set* in a graph G is a set of pairwise non-adjacent vertices. Given a graph G and two independent sets $\mathbf{I}, \mathbf{J}$, imagine that a token (coin) is placed at each vertex of **I**. the *independent set reconfiguration* (ISRECONF) problem asks if one can transform **I** to **J** using a given reconfiguration rule such that all intermediate sets are also independent. The following reconfiguration rules are mainly studied:

- *Token Sliding* (TS rule): A single token can be slid only along an edge of a graph. The ISRECONF problem under TS rule is also known as the SLIDING TOKEN problem.
- *Token Jumping* (TJ rule): A single token can "jump" to any vertex (including non-adjacent one).
- *Token Addition and Removal* (TAR rule): We can either add or remove a single token at

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a time if it results in an independent set of cardinality at least a given threshold.

It is known that ISReconf is PSPACE-complete under any of the three reconfiguration rules for general graphs [5], for planar graphs [1, 4], for perfect graphs [6], and even for bounded bandwidth graphs [8].

Among different variants of ISReconf, the sliding token problem is of particular interest. Given two independent sets $\mathbf{I}$ and $\mathbf{J}$ (having the same cardinality) of a graph G, and imagine that a token (coin) is placed on each vertex in $\mathbf{I}$. Then, the sliding token problem is to determine whether there exists a sequence (called a TS-sequence) $\langle \mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_\ell \rangle$ of independent sets of G such that

(a) $\mathbf{I}_1 = \mathbf{I}$, $\mathbf{I}_\ell = \mathbf{J}$, and $|\mathbf{I}_i| = |\mathbf{I}| = |\mathbf{J}|$ for all i, $1 \leq i \leq \ell$; and

(b) for each i, $2 \leq i \leq \ell$, there is an edge uv in G such that $\mathbf{I}_{i-1} \setminus \mathbf{I}_i = \{u\}$ and $\mathbf{I}_i \setminus \mathbf{I}_{i-1} = \{v\}$, that is, $\mathbf{I}_i$ can be obtained from $\mathbf{I}_{i-1}$ by sliding exactly one token on a vertex $u \in \mathbf{I}_{i-1}$ to its adjacent vertex v along $uv \in E(G)$.

It is well-known that many PSPACE-hardness results can be shown using reduction from sliding token [5]. Beside the PSPACE-completeness of sliding token mentioned above, recently, Kamiński et al. [6] showed that sliding token can be solved in linear time for cographs. Bonsma et al. [2] proved that sliding token can be solved in polynomial time for claw-free graphs. Very recently, Demaine et al. [3] gave a linear-time algorithm for solving sliding token on trees. In their paper, Demaine et al. mentioned that: "The PSPACE-hardness implies that an instance of sliding token may require an exponential number of token-slides even in a minimum-length reconfiguration sequence. In such a case, tokens should make detours to avoid violating to be independent." Because of that, the possibility of "making detours" makes sliding token much more complicated.

In this paper, we extend the idea of Demaine et al. in [3] to develop a polynomial-time algorithm for solving sliding token on unicylic graphs. A unicyclic graph is a connected graph that contains exactly one cycle. The key idea of Demaine et al.'s algorithm is the (linear-time) characterization of what they called *rigid tokens* - the tokens that cannot be moved along any edge of the graph without violating the independence property. On one hand, the structure of unicyclic graphs is very close to trees. More precisely, one can obtained a unicyclic graph from a tree by adding one extra edge. On the other hand, the characterization of rigid tokens in unicyclic graphs is indeed much more complicated (see Lemma 3.1) because of the existence of one single cycle.

2 Preliminaries

2.1 Graph notation

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $|G|$ denote the number of vertices of G. For a vertex $v \in V(G)$, let $N(G,v) = \{w \in V(G) \mid vw \in E(G)\}$ and $N[G,v] = N(G,v) \cup \{v\}$. Similarly, for an arbitrary subset $S \subseteq V(G)$, we write $N[G,S] = \bigcup_{v \in S} N[G,v]$. For a subgraph G' of G, denote by $G - G'$ the subgraph of G induced by $V(G) \setminus V(G')$. Similarly, for a subset $S \subseteq V(G)$, denote by $G - S$ the subgraph of G obtained by removing all vertices in S. For two vertices $u, v \in V(G)$, we denote by $\mathsf{dist}(u,v)$ the number of edges of a shortest uv-path in G.

An independent set $\mathbf{I}$ of a graph G is a subset of $V(G)$ in which for every $u, v \in \mathbf{I}$, uv is not an edge of G. For a subgraph H of G, sometimes we write $\mathbf{I} \cap H$ and $\mathbf{I} - H$ to indicate the sets $\mathbf{I} \cap V(H)$ and $\mathbf{I} \setminus V(H)$, respectively.

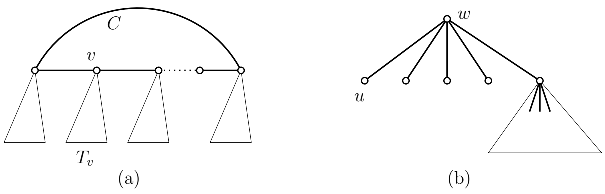

図 1: (a) The tree T_v corresponding to $v \in V(C)$. (b) A safe degree-1 vertex u of a tree.

Let G be a unicyclic graph. Let C be its unique cycle. Let H be the subgraph obtained from G by deleting all edges of C. For each vertex $v \in V(C)$, we define T_v to be the (unique) component of H containing v (see Figure 1(a)). It is not hard to see that each $v \in V(C)$ corresponds to a unique (induced) sugraph (which is indeed a tree) T_v of G. Moreover, any vertex $w \in V(G)$ belongs to exactly one such tree T_v, for some $v \in V(C)$.

For a tree T, a vertex $u \in V(T)$, is called a *safe degree*-1 vertex if $\deg_T(u) = 1$ and its unique neighbor w has exactly one neighbor of degree greater than 1 (see Figure 1(b)). Similarly, for a unicyclic graph G, a vertex u is called a *safe degree*-1 vertex if it is a safe degree-1 vertex in the unique tree T_v containing u, for some vertex $v \in V(C)$.

2.2 Definitions for SLIDING TOKEN

Let $\mathbf{I}$ and $\mathbf{J}$ be two independent sets of a graph G such that $|\mathbf{I}| = |\mathbf{J}|$. If there exists exactly one edge uv in G such that $\mathbf{I} \setminus \mathbf{J} = \{u\}$ and $\mathbf{J} \setminus \mathbf{I} = \{v\}$, then we say that $\mathbf{J}$ can be obtained from $\mathbf{I}$ by *sliding* the token on $u \in \mathbf{I}$ *immediately* to its adjacent vertex v along the edge uv, and denote it by $\mathbf{I} \leftrightarrow \mathbf{J}$, or sometimes by $\mathbf{I} \overset{G}{\leftrightarrow} \mathbf{J}$. Note that "sliding a token" can be reversed, i.e. $\mathbf{I} \leftrightarrow \mathbf{J}$ if and only if $\mathbf{J} \leftrightarrow \mathbf{I}$.

In order to describe the process of sliding tokens, we often use the concept of TS-*sequence.* A TS-*sequence* between two independent sets $\mathbf{I}_1$ and $\mathbf{I}_\ell$ of G is a sequence $\langle \mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_\ell \rangle$ of independent sets of G such that $\mathbf{I}_{i-1} \leftrightarrow \mathbf{I}_i$ for $i = 2, 3, \dots, \ell$. We sometimes write $\mathbf{I} \in \mathcal{S}$ if an independent set $\mathbf{I}$ of G appears in the TS-sequence $\mathcal{S}$. We say that a TS-sequence $\mathcal{S} = \langle \mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_\ell \rangle$ in G *moves* (or *slides*) the token t on vertex $u \in \mathbf{I}_1$ to $v \notin \mathbf{I}_1$ if after apply the sliding steps described in $\mathcal{S}$, the token t is on vertex $v \in \mathbf{I}_\ell$. We write $\mathbf{I}_1 \overset{G}{\leftrightsquigarrow} \mathbf{I}_\ell$ if there exists a TS-sequence $\mathcal{S}$ between $\mathbf{I}_1$ and $\mathbf{I}_\ell$ such that all independent sets $\mathbf{I} \in \mathcal{S}$ satisfy $\mathbf{I} \subseteq V(G)$. Sometimes, to emphasize the existence of a reconfiguration sequence, we also write $\mathbf{I}_1 \underset{\mathcal{S}}{\overset{G}{\leftrightsquigarrow}} \mathbf{I}_\ell$. Moreover, a TS-sequence is *reversible*, i.e. $\mathbf{I}_1 \overset{G}{\leftrightsquigarrow} \mathbf{I}_\ell$ if and only if $\mathbf{I}_\ell \overset{G}{\leftrightsquigarrow} \mathbf{I}_1$. The *length* of a reconfiguration sequence $\mathcal{S}$ is defined as the number of independent sets contained in $\mathcal{S}$.

Assume that a graph G contains distinct components $G_1, G_2, G_3, \dots, G_l$. Let $\mathcal{S} = \langle \mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_\ell \rangle$ be a TS-sequence in G that reconfigures $\mathbf{I}_1$ to $\mathbf{I}_\ell$. Note that if $\mathbf{I}$ is an independent set of G then $\mathbf{I} \cap G_i$ $(1 \leq i \leq l)$ is also an independent set of G_i. Therefore, $\mathcal{S}$ can be *restricted* to a TS-sequence $\mathcal{S}_i = \langle \mathbf{I}_1 \cap G_i, \dots, \mathbf{I}_\ell \cap G_i \rangle$ in G_i that reconfigures $\mathbf{I}_1 \cap G_i$ to $\mathbf{I}_\ell \cap G_i$. Conversely, if $\mathcal{S}_i = \langle \mathbf{I}^i_1, \dots, \mathbf{I}^i_p \rangle$ is a TS-sequence in G_i that reconfigures $\mathbf{I}^i_1$ to $\mathbf{I}^i_p$, and if $\mathbf{I}$ is any independent set of G such that $\mathbf{I}^i_1 \subseteq \mathbf{I}$, then $\mathcal{S}_i$ can be *extended* to a TS-sequence $\mathcal{S} = \langle \mathbf{I}^i_1 \cup (\mathbf{I} \setminus \mathbf{I}^i_1), \dots, \mathbf{I}^i_p \cup (\mathbf{I} \setminus \mathbf{I}^i_1) \rangle$ in G that reconfigures $\mathbf{I} = \mathbf{I}^i_1 \cup (\mathbf{I} \setminus \mathbf{I}^i_1)$ to some independent set $\mathbf{I}' = \mathbf{I}^i_p \cup (\mathbf{I} \setminus \mathbf{I}^i_1)$ of G. Note that $\mathcal{S}$ involves only sliding tokens on G_i. This observation will be implicitly used in many statements of this paper.

3 Sliding tokens on unicyclic graphs

3.1 Rigid tokens

Let $\mathbf{I}$ be an independent set of a graph G. Let t be a token placed at vertex $u \in \mathbf{I}$. We say that t is $(G, \mathbf{I})$-*rigid* if for every independent set $\mathbf{I}'$ such that $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{I}'$, $u \in \mathbf{I}'$. If t is not $(G, \mathbf{I})$-rigid, we say that it is $(G, \mathbf{I})$-*movable*. Similarly, for a subgraph H of G, we say that a token t on $v \in \mathbf{I} \cap H$ is $(H, \mathbf{I} \cap H)$-rigid if and only if for every independent set $\mathbf{J}$ such that $\mathbf{I} \cap H \overset{H}{\leftrightsquigarrow} \mathbf{J}$, $v \in \mathbf{J}$. For an independent set $\mathbf{I}$ of G, denote by $\mathsf{R}(\mathbf{I})$ the set of all vertices when $(G, \mathbf{I})$-rigid tokens are placed.

First of all, we characterize the property of $(G, \mathbf{I})$-rigid tokens in a unicyclic graph G.

Lemma 3.1. *Let $\mathbf{I}$ be an independent set of a unicyclic graph G. Assume that the unique cycle C of G is of length k $(3 \leq k \leq |G|)$. For any vertex $u \in \mathbf{I}$, the token t on u is $(G, \mathbf{I})$-rigid if and only if for every vertex $v \in N(G, u)$, there exists a vertex $w \in \big(N(G, v) \setminus \{u\}\big) \cap \mathbf{I}$ satisfying one of the following conditions:*

(i) The token t_w on w is $(G', \mathbf{I} \cap G')$-rigid, where $G' = G - N[G, u]$. (see Figure 2.)

(ii) $u \notin V(C)$, $\{v, w\} \subseteq V(C)$, the token t_w on w is not $(G', \mathbf{I} \cap G')$-rigid, and for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$. (see Figure 3.)

Proof. We first show the if-part. Assume that for any $v \in N(G, u)$, either (i) or (ii) holds. We want to show that t is indeed $(G, \mathbf{I})$-rigid. Assume for the contradiction that there exists a TS-sequence $\mathcal{S}$ in G that moves t to v, i.e. the token t on u is not $(G, \mathbf{I})$-rigid.

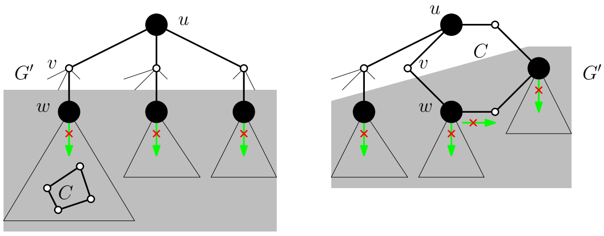

図 2: Case (i) of Lemma 3.1. The token on u is $(G, \mathbf{I})$-rigid.

- If (i) holds, that is, there exists a vertex $w \in \big(N(G, v) \setminus \{u\}\big) \cap \mathbf{I}$ such that the token t_w on w is $(G', \mathbf{I} \cap G')$-rigid, where $G' = G - N[G, u]$. Since the TS-sequence $\mathcal{S}$ (in G) moves t to v, it is necessary that $\mathcal{S}$ moves t_w to a vertex in $N(G, w) \setminus \{v\} = N(G', w)$ first. (On the other hand, moving t_w to a vertex in $N(G, w) \setminus \{v\}$ does not guarantee that t can be moved to v.) Note that before moving t, any TS-sequence $\mathcal{S} = \langle \mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_\ell \rangle$ in G can be restricted to a TS-sequence $\mathcal{S}' = \langle \mathbf{I}_1 \setminus \{u\}, \mathbf{I}_2 \setminus \{u\}, \ldots, \mathbf{I}_\ell \setminus \{u\} \rangle = \langle \mathbf{I}_1 \cap G', \mathbf{I}_2 \cap G', \ldots, \mathbf{I}_\ell \cap G' \rangle$ in G', and vice versa. It follows that there exists a TS-sequence $\mathcal{S}'$ in G' (restricted from $\mathcal{S}$) that moves t_w to a vertex in $N(G', w)$, which contradicts to the assumption that t_w is $(G', \mathbf{I} \cap G')$-rigid.

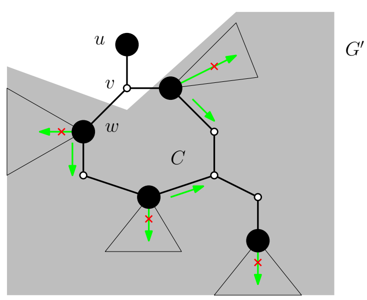

図 3: Case (ii) of Lemma 3.1. The token on u is $(G, \mathbf{I})$-rigid.

- If (ii) holds, that is, there exists a vertex $w \in \big(N(G,v) \setminus \{u\}\big) \cap \mathbf{I}$ satisfying that $u \notin V(C)$, $\{v,w\} \subseteq V(C)$, the token t_w on w is not $(G', \mathbf{I} \cap G')$-rigid, and for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, the path $P = C - v$ (which, in this case, is a subgraph of G') satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$. Assume that $P = p_1 p_2 \dots p_{k-1}$. Since $w \in N(G,v)$, $w \in V(C)$ and $P = C - v$, we can assume that $p_1 = w$. Since for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, the path P satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$, it follows that there does not exist any TS-sequence in G' that moves a token in $P \cap \mathbf{I} \subseteq \mathbf{I} \cap G'$ to a vertex in $G' - P$, or moves a token in $G' - P$ to a vertex in P. Moreover, if a token in $P \cap \mathbf{I}'$ is $(P, P \cap \mathbf{I}')$-rigid then so is every token in $P \cap \mathbf{I}'$. Consequently, for any such $\mathbf{I}'$, $\{p_1, p_{k-1}\} \cap \mathbf{I}' \neq \emptyset$. In other words, any TS-sequence in G' that moves t_w from $w = p_1$ to p_2 also moves a different token in P to p_{k-1} (if there is no token on it). Since $P = C - v$, it follows that $\{p_1, p_{k-1}\} \subseteq N(C,v)$, and therefore $N(C,v) \cap \mathbf{I}' \neq \emptyset$, where $\mathbf{I}'$ is such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$. As mentioned above, before moving t, any TS-sequence $\mathcal{S}' = \langle \mathbf{I}'_1, \mathbf{I}'_2, \dots, \mathbf{I}'_q \rangle$ in G' can be extended to a TS-sequence $\mathcal{S} = \langle \mathbf{I}'_1 \cup \{u\}, \mathbf{I}'_2 \cup \{u\}, \dots, \mathbf{I}'_q \cup \{u\} \rangle$ in G, and vice versa. Therefore, $N(C,v) \cap \mathbf{J} \neq \emptyset$ for every $\mathbf{J}$ such that $\mathbf{I} \underset{\mathcal{S}}{\overset{G}{\leftrightsquigarrow}} \mathbf{J}$, where $\mathcal{S}$ does not involve sliding t, which contradicts to our assumption that t can be slid to v.

Next, we show the only-if-part. Assume that t is $(G, \mathbf{I})$-rigid. We want to show that either (i) or (ii) holds. Assume for the contradiction that both (i) and (ii) do not hold, that is, for any $u \in \mathbf{I}$, there exists a vertex $v \in N(G,u)$ such that either $\big(N(G,v) \setminus \{u\}\big) \cap \mathbf{I} = \emptyset$ or for every $w \in \big(N(G,v) \setminus \{u\}\big) \cap \mathbf{I}$, the token t_w on w is not $(G', \mathbf{I} \cap G')$-rigid, and one of the following conditions does not hold:

(a) $u \notin V(C)$;
(b) $\{v,w\} \subseteq V(C)$; and
(c) for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$.

We claim that in both cases, there exists a TS-sequence $\mathcal{S}$ in G that moves t to v. If $\big(N(G,v) \setminus \{u\}\big) \cap \mathbf{I} = \emptyset$, one can slide t to v immediately. We now consider the case when $\big(N(G,v) \setminus \{u\}\big) \cap \mathbf{I} \neq \emptyset$. Assume that for every $w \in \big(N(G,v) \setminus \{u\}\big) \cap \mathbf{I}$, the token t_w on w is not $(G', \mathbf{I} \cap G')$-rigid, and one of the conditions (a), (b) and (c) does not hold. Note that by definition, $w \neq u$.

Since t_w is not $(G', \mathbf{I} \cap G')$-rigid, there exists a TS-sequence $\mathcal{S}'$ in G' that moves t_w to a vertex in $N(G',w)$. If $w \notin V(C)$, the (connected) component H of G' containing w does not contain C. Moreover, since G contains exactly one cycle, $V(H) \cap N(G,v) = \{w\}$. Therefore, $\mathcal{S}'$ does not move any token to a vertex in $N(G,v) \setminus \{u\}$, and hence t can be slid to v in G. If $w \in V(C)$, consider the following cases:

- Case 1: $u \in V(C)$ and $w \notin V(C)$. This case cannot happen since G contains exactly one cycle.
- Case 2: $u \notin V(C)$ and $v \notin V(C)$. Since $v \notin V(C)$, the path $P = C - v$ cannot be defined, and then condition (c) does hot happen, which means that one can slide t_w to a vertex in $N(G',w)$ without moving any token to a vertex in $N(C,v)$ during the sliding process. Then, t can be slid to v.
- Case 3: $u \notin V(C)$ and $v \in V(C)$. In this case, both conditions (a) and (b) hold. Therefore, (c) does not hold, which means that either $|P \cap \mathbf{I}| < \lfloor k/2 \rfloor$ or one can slide a token (which is not necessarily t_w) in $P \cap \mathbf{I}$ to a vertex in $G' - P$, which also decreases the

number of tokens in P. This means that t_w can now be slid to a vertex in $N(G', w)$ without moving any token to a vertex in $N(C, v)$ during the sliding process. Then, t can be slid to v.

- Case 4: $u \in V(C)$ and $v \in V(C)$. Note that in this case, the path $P = C - v = p_1 p_2 \dots p_{k-1}$ is not a subgraph of G' and now contains both $w = p_1$ and $u = p_{k-1}$ as its endvertices. Also, since both u and w are in $\mathbf{I}$, C must be of length $k \geq 4$. On the other hand, note that if P satisfies the condition (c), then the path $P' = P \cap G' = P - N[G, u] = p_1 p_2 \dots p_{k-3}$ also satisfies (c). If (c) does not hold, we can use the same argument as in the previous part (replacing P by P') to show that t can be slid to v. On the other hand, if (c) holds, then since t_w is not $(G', \mathbf{I} \cap G')$-rigid, one can slide t_w to $N(P', w)$ and moves a token to p_{k-3} during the sliding process. Since $p_{k-3} \notin N(G, u)$, this sliding process can be applied in G, and hence t can now be slid to v.

□

Next, we claim that the condition (ii) of Lemma 3.1 indeed can be checked in polynomial time.

Lemma 3.2. *Given a unicyclic graph G. Assume that the unique cycle C of G is of length k ($3 \leq k \leq n$). Let $\mathbf{I}$ be an independent set of G. Let $u \in \mathbf{I}$, $v \in N(G, u)$ and $w \in \big(N(G, v) \setminus \{u\}\big) \cap \mathbf{I}$. Assume that $u \notin V(C)$, $\{v, w\} \subseteq V(C)$, and the token t_w on w is not $(G', \mathbf{I} \cap G')$-rigid, where $G' = G - N[G, u]$. Then, one can check in polynomial time that for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$.*

Proof. First of all, we claim that for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, P satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$ if and only if

(a) $|P \cap \mathbf{I}| = \lfloor k/2 \rfloor$; and

(b) for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, any token on $x \in P \cap \mathbf{I}'$ is $(T_x, \mathbf{I}' \cap T_x)$-rigid.

It is clear from the definition of rigid tokens that if both (a) and (b) hold then for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$.

Now, if for any independent set $\mathbf{I}'$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, P satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$, then (a) clearly holds since $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I} \cap G'$. Assume that (b) does not hold, that is, there exists an independent set $\mathbf{I}'$ of G', $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'$, and a token t_x on $x \in P \cap \mathbf{I}'$ such that t_x is not $(T_x, \mathbf{I}' \cap T_x)$-rigid. It follows that there exists a TS-sequence $\mathcal{S}_x = \langle \mathbf{I}' \cap T_x = \mathbf{I}^x_1, \dots, \mathbf{I}^x_q \rangle$ in T_x that moves t_x to a vertex in $N(T_x, x)$. Since $\mathbf{I}^x \cup \big(\mathbf{I}' - T_x\big)$ forms an independent set of G', where $\mathbf{I}^x$ is an independent set in T_x, $\mathcal{S}_x$ can be extended to a TS-sequence $\mathcal{S} = \langle \mathbf{I}', \dots, \mathbf{I}^x_q \cup \big(\mathbf{I}' - T_x\big) \rangle$ in G'. Hence, $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}' \overset{G'}{\leftrightsquigarrow} \mathbf{I}^x_q \cup \big(\mathbf{I}' - T_x\big)$. Note that $\mathcal{S}_x$ (and hence $\mathcal{S}$) only involves sliding tokens in T_x. Therefore, we have that $\big|P \cap \big(\mathbf{I}^x_q \cup \big(\mathbf{I}' - T_x\big)\big)\big| = \lfloor k/2 \rfloor - 1$, which is a contradiction. Hence, (b) must hold.

We now only need to check if conditions (a) and (b) hold. The condition (a) can obviously be checked in $O(1)$ time.

From the proof of Lemma 3.1, we know that there does not exist any TS-sequence in G' that moves a token in $P \cap \mathbf{I} \subseteq \mathbf{I} \cap G'$ to a vertex in $G' - P$, or moves a token in $G' - P$ to a vertex in P. Moreover, if a token in $P \cap \mathbf{I}'$ is $(P, P \cap \mathbf{I}')$-rigid then so is every token in $P \cap \mathbf{I}'$. It follows that k must indeed be odd. We claim that there exist two independent sets $\mathbf{I}'_1$ and $\mathbf{I}'_2$ of G' such that $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'_1$, $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{I}'_2$, $(P \cap \mathbf{I}'_1) \cup (P \cap \mathbf{I}'_2) = V(P)$ and $(P \cap \mathbf{I}'_1) \cap (P \cap \mathbf{I}'_2) = \emptyset$. In order to construct $\mathbf{I}'_1$ and $\mathbf{I}'_2$, an important assumption we need to keep in mind is that conditions (a) and (b) must hold for $\mathbf{I} \cap G'$ in the first place.

Let $P = p_1 p_2 \ldots p_{k-1}$, and note that k is odd, then one can define $P \cap \mathbf{I}'_1 = \{p_1, p_3, \ldots, p_{k-2}\}$ and $P \cap \mathbf{I}'_2 = \{p_2, p_4, \ldots, p_{k-1}\}$. $\mathbf{I}'_1$ (and similarly, $\mathbf{I}'_2$) can be obtained from $\mathbf{I} \cap G'$ using token sliding as follows. Let i be the smallest index such that $p_i \in \big(\mathbf{I} \cap G'\big) \setminus \mathbf{I}'_1$. From the definition of $P \cap \mathbf{I}'_1$, i must be even. By condition (a), it follows that $p_j \in \mathbf{I}'_1$ for j odd, $j < i-1$, and $p_j \in \big(\mathbf{I} \cap G'\big) \setminus \mathbf{I}'_1$ for j even, $j \geq i$. Additionally, by condition (b), the token t_{p_i} on p_i can only be slid to p_{i-1}, which means that there exists a TS-sequence $\mathcal{S}_{p_i}$ in G' that slides t_{p_i} to p_{i-1}. Repeat the described steps until all tokens on vertices in $P \cap \mathbf{I}$ are slid to vertices in $P \cap \mathbf{I}'_1$. Since G' is a forest, each such $\mathcal{S}_{p_i}$ described above can be constructed in at most $O(|G'|^2)$ time (see [3, Theorem 2]). Hence, $\mathbf{I}'_1$ and $\mathbf{I}'_2$ can be constructed in $O(|G'|^2)$ time.

From the definition of rigid tokens and the above arguments, in order to check if (b) holds, it is enough to check if (b) holds for the cases $\mathbf{I}' = \mathbf{I} \cap G$, $\mathbf{I}' = \mathbf{I}'_1$ and $\mathbf{I}' = \mathbf{I}'_2$, which can be done in at most $O(|P \cap \mathbf{I}'|.|G'|)$ time (see [3, Theorem 1]). In total, the checking process takes at most $O(n^2)$ time. $\square$

Now, we claim that it can be decided in polynomial time whether the token on u is $(G, \mathbf{I})$-rigid, for any given unicyclic graph G.

Lemma 3.3. *Given a unicyclic graph G with n vertices. Assume that the unique cycle C of G is of length k ($3 \leq k \leq n$). Let $\mathbf{I}$ be an independent set of G and let $u \in \mathbf{I}$. Then, it can be decided in polynomial time whether the token on u is $(G, \mathbf{I})$-rigid.*

Proof. We claim that Algorithm 1 can be used to decide in polynomial time whether the token on u is $(G, \mathbf{I})$-rigid.

The correctness of Algorithm 1 clearly follows

Algorithm 1 Check if a token on $u \in \mathbf{I}$ is $(G, \mathbf{I})$-rigid.

Input: A unicyclic graph G, its unique cycle C of length k, an independent set $\mathbf{I}$ of G, and a vertex $u \in \mathbf{I}$.

Output: Return YES if the token on u is $(G, \mathbf{I})$-rigid; otherwise, return NO.

function CheckRigid(G, $\mathbf{I}$, u)
  $G' = G - N[G, u]$
  for all $v \in N(G, u)$ **do**
    if $\big(N(G, v) \setminus \{u\}\big) \cap \mathbf{I} = \emptyset$ **then return** NO
    else
      for all $w \in \big(N(G, v) \setminus \{u\}\big) \cap \mathbf{I}$ **do**
        if CheckRigid(G', $\mathbf{I} \cap G'$, w) = NO **then**
          if $u \notin V(C)$, $v \in V(C)$, $w \in V(C)$ **then**
            if Lemma 3.2 does not hold **then return** NO
            end if
          else
            return NO
          end if
        end if
      end for
      return YES
    end if
  end for
end function

from Lemma 3.1 and Lemma 3.2. We claim that its running time is at most $O(n^2)$ time. Note that every components of G' are trees, except the one that contains C (if exists). Moreover, observe that if G' contains (connected) components $G'_1, \ldots, G'_r$ then each TS-sequence in G' can be restricted to each component G'_i $(i = 1, 2, \ldots, r)$, and vice versa. Let H be the component of G' containing w. If H is a tree, then $\textsc{CheckRigid}(G', \mathbf{I} \cap G', w) = \textsc{CheckRigid}(H, \mathbf{I} \cap H, w)$ takes at most $O(|H|)$ time (see [3, Lemma 2]). On the other hand, if H contains C, since the checking step in line 9 takes at most $O(n^2)$ time as described in Lemma 3.2, the function $\textsc{CheckRigid}(G', \mathbf{I} \cap G', w)$ takes at most $O(n^2)$ time. In total, Algorithm 1 takes at most $O(n^2)$ time. □

The next lemma is useful in showing the correctness of our algorithm for solving SLIDING TOKEN on unicyclic graphs.

Lemma 3.4. *Let $\mathbf{I}$ be an independent set of a unicylic graph G. Let C be the unique cycle of G. Assume that C is of length k. Assume that all tokens are $(G, \mathbf{I})$-movable. Let $v \in V(G)$ be such that $v \notin \mathbf{I}$. Then, there exists at most one neighbor $w \in N(G, v) \cap \mathbf{I}$ such that the token on w is $(G'', \mathbf{I} \cap G'')$-rigid, where $G'' = G - v$. Moreover, if both v and w are in $V(C)$ and for any independent set $\mathbf{I}'$ of G'' such that $\mathbf{I} \cap G'' \overset{G''}{\rightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$, then the token on any vertex in $N(G, v) \cap \mathbf{I}$ is not $(G'', \mathbf{I} \cap G'')$-rigid.*

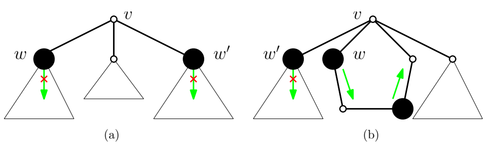

図 4: Illustration for Lemma 3.4.

Proof. Assume for the contradiction that v has two distinct neighbors w and w' in $N(G, v) \cap \mathbf{I}$ such that the token t_w on w and the token $t_{w'}$ on w' are both $(G'', \mathbf{I} \cap G'')$-rigid (see Figure 4(a)). Since t_w is $(G'', \mathbf{I} \cap G'')$-rigid, it cannot be slid to any vertex in $N(G'', w)$. On the other hand, t_w is $(G, \mathbf{I})$-movable, hence the only way to slide t_w out of w is to move t_w to $v \in V(G) \setminus V(G'')$. But now, since $w' \in N(G, v)$, we need to slide $t_{w'}$ to a vertex in $N(G'', w')$ first. But this is a contradiction, since $t_{w'}$ is $(G'', \mathbf{I} \cap G'')$-rigid.

Now, suppose that both v and w are in $V(C)$ and for any independent set $\mathbf{I}'$ of G'' such that $\mathbf{I} \cap G'' \overset{G''}{\rightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$. Assume for the contradiction that there exists a vertex w' in $N(G, v) \cap \mathbf{I}$ such that the token $t_{w'}$ on w' is $(G'', \mathbf{I} \cap G'')$-rigid (see Figure 4(b)). If k is even, since for any independent set $\mathbf{I}'$ of G'' such that $\mathbf{I} \cap G'' \overset{G''}{\rightsquigarrow} \mathbf{I}'$, the path $P = C - v$ satisfies $|P \cap \mathbf{I}'| = \lfloor k/2 \rfloor$, it follows that all tokens on $P \cap \mathbf{I}$ are $(G'', \mathbf{I} \cap G'')$-rigid. Hence, there are two neighbors of v in C satisfying that the tokens on them are all $(G'', \mathbf{I} \cap G'')$-rigid, which contradicts to the previous part of our proof. Therefore, k must be odd and then, no token on $P \cap \mathbf{I}$ is $(G'', \mathbf{I} \cap G'')$-rigid. Since $t_{w'}$ is $(G'', \mathbf{I} \cap G'')$-rigid, $w' \notin V(C)$ and $t_{w'}$ cannot be slid to any vertex in $N(G'', w')$. Since $t_{w'}$ is $(G, \mathbf{I})$-movable, the only way to slide $t_{w'}$ out of w' is to move $t_{w'}$ to $v \in V(G) \setminus V(G'')$. By Lemma 3.1(ii), $t_{w'}$ is $(G, \mathbf{I})$-rigid, which is a contradiction. Hence, the token on any vertex in $N(G, v) \cap \mathbf{I}$ is not $(G'', \mathbf{I} \cap G'')$-rigid. □

3.2 Algorithm

Using the same algorithm provided by Demaine et al. [3], we can solve the SLIDING TOKEN problem on unicyclic graphs in polynomial time.

Let G be a unicyclic graph with n vertices, and let $\mathbf{I}$ and $\mathbf{J}$ be two given independent sets of G.

Step 1. Compute $\mathsf{R}(\mathbf{I})$ and $\mathsf{R}(\mathbf{J})$ using Lemma 3.3. If $\mathsf{R}(\mathbf{I}) \neq \mathsf{R}(\mathbf{J})$, then return NO; otherwise go to Step 2.

Step 2. Delete the vertices in $N[G, \mathsf{R}(\mathbf{I})] = N[G, \mathsf{R}(\mathbf{J})]$ from G, and obtain a subgraph F consisting of q connected components $G_1, G_2, \dots, G_q$. If $|\mathbf{I} \cap G_j| = |\mathbf{J} \cap G_j|$ holds for every $j \in \{1, 2, \dots, q\}$, then return YES; otherwise return NO.

By Lemma 3.3 we can determine whether one token in an independent set $\mathbf{I}$ of G is $(G, \mathbf{I})$-rigid or not in $O(n^3)$ time, and hence Step 1 can be done in time $O(n^2) \times (|\mathbf{I}| + |\mathbf{J}|) = O(n^3)$. Clearly, Step 2 can be done in $O(n)$ time. Therefore, the described algorithm runs in $O(n^3)$ time in total. In the remaining part of this section, we show the correctness of the above algorithm.

First of all, we show the correctness of **Step 1**. In the next lemma, we show that if the set of $(G, \mathbf{I})$-rigid and $(G, \mathbf{J})$-rigid tokens are different, then $\mathbf{I}$ cannot be reconfigured to $\mathbf{J}$.

Lemma 3.5. *[3, Lemma 5] Suppose that for two independent sets* $\mathbf{I}$, $\mathbf{J}$ *of a unicyclic graph* G, $\mathsf{R}(\mathbf{I}) \neq \mathsf{R}(\mathbf{J})$. *Then, there does not exist any* TS*-sequence* $\mathcal{S}$ *such that* $\mathbf{I} \overset{G}{\underset{\mathcal{S}}{\leftrightsquigarrow}} \mathbf{J}$.

Next, we show the correctness of **Step 2**. We start by showing that in case the set of $(G, \mathbf{I})$-rigid and $(G, \mathbf{J})$-rigid tokens are the same, removing all rigid tokens and its neighbors does not affect the final answer of the SLIDING TOKEN problem.

Lemma 3.6. *[3, Lemma 6] Suppose that* $\mathsf{R}(\mathbf{I}) = \mathsf{R}(\mathbf{J})$ *for two given independent sets* $\mathbf{I}$ *and* $\mathbf{J}$ *of a an unicyclic graph* G*, and let* G' *be the graph obtained from* G *by deleting the vertices in* $N[G, \mathsf{R}(\mathbf{I})] = N[G, \mathsf{R}(\mathbf{J})]$ *from* G*. Then* $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{J}$ *if and only if* $\mathbf{I} \cap G' \overset{G'}{\leftrightsquigarrow} \mathbf{J} \cap G'$*. Furthermore, all tokens in* $\mathbf{I} \cap G'$ *are* $(G', \mathbf{I} \cap G')$*-movable, and all tokens in* $\mathbf{J} \cap G'$ *are* $(G', \mathbf{J} \cap G')$*-movable.*

If $\mathsf{R}(\mathbf{I}) = \mathsf{R}(\mathbf{J}) = \emptyset$ for any two given independent sets $\mathbf{I}$, $\mathbf{J}$ of a unicyclic graph G, we claim that $\mathbf{I}$ can be reconfigured to $\mathbf{J}$ using TS rule if and only if $|\mathbf{I}| = |\mathbf{J}|$.

Lemma 3.7. *Let* G *be a unicyclic graph. Let* $\mathbf{I}$ *and* $\mathbf{J}$ *be two given independent sets of* G*. Assume that there are no* $(G, \mathbf{I})$*-rigid and* $(G, \mathbf{J})$*-rigid tokens. Then* $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{J}$ *if and only if* $|\mathbf{I}| = |\mathbf{J}|$.

Before proving Lemma 3.7, we show some extra claims. From now on, we assume that for any independent set $\mathbf{I}$ of a unicyclic graph G, the token on any $u \in \mathbf{I}$ is $(G, \mathbf{I})$-movable. First of all, we claim that Lemma 3.7 holds when G is a cycle.

Lemma 3.8. *Let* C *be a cycle. Let* $\mathbf{I}$ *and* $\mathbf{J}$ *be two given independent sets of* C*. Assume that there are no* $(C, \mathbf{I})$*-rigid and* $(C, \mathbf{J})$*-rigid tokens. Then* $\mathbf{I} \overset{C}{\leftrightsquigarrow} \mathbf{J}$ *if and only if* $|\mathbf{I}| = |\mathbf{J}|$.

Proof. If $\mathbf{I} \overset{C}{\leftrightsquigarrow} \mathbf{J}$ then clearly $|\mathbf{I}| = |\mathbf{J}|$. Now, assume that $|\mathbf{I}| = |\mathbf{J}|$. We claim that $\mathbf{I} \overset{C}{\leftrightsquigarrow} \mathbf{J}$. Let $C = v_1 v_2 \dots v_k v_1$. Let $\mathbf{I}'$ be an independent set of C such that $|\mathbf{I}'| = |\mathbf{I}| = |\mathbf{J}| \leq \lfloor k/2 \rfloor$ and $v_i \in \mathbf{I}'$ if i is odd. We claim that $\mathbf{I} \overset{C}{\leftrightsquigarrow} \mathbf{I}'$, and similarly, $\mathbf{J} \overset{C}{\leftrightsquigarrow} \mathbf{I}'$. Consider the following cases:

- *Case 1:* $|\mathbf{I}| = \lfloor k/2 \rfloor$. If k is even then $\mathbf{I} = \mathbf{I}'$ because there are no $(C, \mathbf{I})$-rigid tokens. If k is odd, let i be the smallest index such that $v_i \in \mathbf{I} \setminus \mathbf{I}'$, $2 \leq i \leq k$. Hence, from the definition of $\mathbf{I}'$, i must be even. Moreover, $v_j \in \mathbf{I}'$ for odd j, $1 \leq j < i - 1$, and $v_j \in \mathbf{I}$ for even j, $i \leq j \leq k - 1$. Hence, one can slide the token on v_i to $v_{i-1} \in \mathbf{I}' \setminus \mathbf{I}$, then slide the token on v_{i+2} to v_{i+1}, and so on. Let $\mathcal{S}$ be the TS-sequence describing the above process, then clearly $\mathbf{I} \overset{C}{\underset{\mathcal{S}}{\leftrightsquigarrow}} \mathbf{I}'$, since each sliding step reduces $|\mathbf{I}' \setminus \mathbf{I}|$.

- *Case 2:* $|\mathbf{I}| < \lfloor k/2 \rfloor$. Let i be the smallest index such that $v_i \in \mathbf{I} \setminus \mathbf{I}'$, $2 \le i \le k$. If $i = 2$ then since there are no $(C, \mathbf{I})$-rigid tokens, we can assume without loss of generality that $v_k \notin \mathbf{I}$; otherwise there exists a TS-sequence that slides the token in v_k to v_{k-1} and then one can deal with the resulting independent set. Let j be the smallest index such that $v_j \in \mathbf{I}' \setminus \mathbf{I}$, $1 \le j \le k$. Since $v_i \notin \mathbf{I}'$, $i > j$. Now, one can slide v_i to v_j and repeat the process. Let $\mathcal{S}$ be the TS-sequence describing the above process, then clearly $\mathbf{I} \overset{C}{\underset{\mathcal{S}}{\leftrightsquigarrow}} \mathbf{I}'$.

□

Next, we claim that there exists a TS-sequence that slides a token to a given safe degree-1 vertex v of G.

Lemma 3.9. *Let G be a unicyclic graph having at least one vertex of degree 1. Assume that the unique cycle C of G is of length k. Let $\mathbf{I}$ be an independent set of G. Let $v \in \mathbf{I}$ be a safe degree-1 vertex of G. Assume that all tokens are $(G, \mathbf{I})$-movable. Then, there exists an independent set $\mathbf{I}'$ satisfying that $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{I}'$ and $v \in \mathbf{I}'$.*

Proof. The proof of this lemma is similar to the proof of [3, Lemma 8], except the process of choosing the token which can be used to slide to v.

Suppose that $v \notin \mathbf{I}$; otherwise the lemma clearly holds. We will show that one of the closest tokens from v can be slid to v. Let $M = \{w \in \mathbf{I} \mid \mathsf{dist}(v, w) = \min_{x \in \mathbf{I}} \mathsf{dist}(v, x)\}$. Let w be an arbitrary vertex in M, and let $P = (p_0 = v, p_1, \ldots, p_\ell = w)$ be a shortest vw-path in G. If $\ell = 1$ and hence $p_1 \in \mathbf{I}$, then we can simply slide the token on p_1 to v. Thus, we may assume that $\ell \ge 2$.

We note that no token is placed on the vertices $p_0, \ldots, p_{\ell-1}$ and the neighbors of $p_0, \ldots, p_{\ell-2}$, because otherwise the token on w is not closest to v. Let $M' = M \cap N(G, p_{\ell-1})$. Consider the following cases:

- *Case 1:* $p_{\ell-1} \notin V(C)$. Since $p_{\ell-1} \notin \mathbf{I}$, by Lemma 3.4 there exists at most one vertex $w' \in M'$ such that the token on w' $(G'', \mathbf{I} \cap G'')$-rigid, where $G'' = G - p_{\ell-1}$. We choose such a vertex w' if exists, otherwise choose an arbitrary vertex in M' and regard it as w'.
- *Case 2:* $p_{\ell-1} \in V(C)$. Since $p_{\ell-1} \in V(C)$, by Lemma 3.4, if for any independent set $\mathbf{J}$ of G'' such that $\mathbf{I} \cap G'' \overset{G''}{\leftrightsquigarrow} \mathbf{J}$, the path $C - p_{\ell-1}$ satisfies $|(C - p_{\ell-1}) \cap \mathbf{J}| = \lfloor k/2 \rfloor$, then the token on any vertex in $N(G, p_{\ell-1}) \cap \mathbf{I}$ is not $(G'', \mathbf{I} \cap G'')$-rigid, and we choose such a vertex in C as w', otherwise choose an arbitrary vertex in M' and regard it as w'.

Since all tokens on the vertices w'' in $M' \setminus \{w'\}$ are $(G'', \mathbf{I} \cap G'')$-movable, we first slide the tokens on w'' to some vertices in the component of G'' containing w''. Then, we can slide the token on w' to v along the path P. In this way, we can obtain an independent set $\mathbf{I}'$ such that $v \in \mathbf{I}'$ and $\mathbf{I} \overset{T}{\leftrightsquigarrow} \mathbf{I}'$. □

Lemma 3.10. *Let G be a unicyclic graph containing at least one vertex of degree 1. Let $\mathbf{I}$ be an independent set of G such that $\mathbf{I}$ contains a safe degree-1 vertex v of G. Assume that all tokens in G are $(G, \mathbf{I})$-movable. Then, all tokens in G^* are $(G^*, \mathbf{I}^*)$-movable, where $\mathbf{I}^* = \mathbf{I} \setminus \{v\}$ and G^* is the graph obtained from G by removing v, its unique neighbor u, and all resulting islated vertices.*

Proof. Suppose for the contradiction that there exists a token on a vertex in $\mathbf{I}^*$ which is $(G^*, \mathbf{I}^*)$-rigid. Let w be such vertex which is closest to v. Let P be a shortest path between v and w. Let z be the unique neighbor of w in P. Assume that $N(G, z) = \{w_1, w_2, \ldots, w_p\}$, where $w_1 \in V(P)$ and $w_p = w$.

First of all, we claim some useful facts.

- Since $\deg_G(v) = 1$, $v \notin V(C)$.

- Since v is $(G, \mathbf{I})$-movable, it follows that there are no token on any degree-1 neighbor of u other than v.
- Since $(\{v\}, \emptyset)$ is a single component of $G-u$, the token on v is $(G-u, \mathbf{I} \cap (G-u))$-rigid. More generally, if a token is $(H, \mathbf{I} \cap H)$-rigid (or $(H, \mathbf{I} \cap H)$-movable) for any connected component H of G, then it is also $(G, \mathbf{I} \cap G)$-rigid (or $(G, \mathbf{I} \cap G)$-movable). Conversely, if a token in G is $(G, \mathbf{I} \cap G)$-rigid (or $(G, \mathbf{I} \cap G)$-movable) and that token is placed at a vertex belonging to H, then it is $(H, \mathbf{I} \cap H)$-rigid (or $(H, \mathbf{I} \cap H)$-movable). This follows from the observation we made at the end of Section 2.2.
- Since the token t_p on w_p is $(G^*, \mathbf{I}^*)$-rigid, then regardless of whether $z \in V(C)$, it must be at least $(G^* - z, \mathbf{I} \cap (G^* - z))$-rigid because otherwise one can slide t_p (along edges of G^*) to a vertex in $N(G^* - z, w_p) = N(G^*, w_p) \setminus \{z\}$.

Consider the following cases:

- *Case* 1*:* $z = u$. Since all tokens placed at vertices of $\mathbf{I}$ are $(G, \mathbf{I})$-movable, $u \notin \mathbf{I}$ and one neighbor v of u is $(G-u, \mathbf{I} \cap (G-u))$-rigid, by Lemma 3.4, the token t_p on w_p must be $(G-u, \mathbf{I} \cap (G-u))$-movable, which is a contradiction since t_p is $(G^*, \mathbf{I}^*)$-rigid, G^* is a connected component of $G-u$ containing w_p, and $\mathbf{I}^* = (\mathbf{I} \cap (G-u)) \setminus \{v\}$.
- *Case* 2*:* $z \neq u$.
  - *Case* 2-1*:* $z \in V(C)$.

    Since $z \in V(C)$, $G-z$ is a forest. Moreover, v is a safe degree-1 vertex of $G-z$, hence by [3, Lemma 9], every token in $G-u-z$ except the token on v must be $(G-u-z, \mathbf{I} \cap (G-u-z))$-movable. On the other hand, since $G^* - z$ is a component of $G-u-z$, every $(G^*, \mathbf{I} \cap G^*)$-rigid tokens is also $(G-u-z, \mathbf{I} \cap (G-u-z))$-rigid. Moreover, since t_p is $(G^* - z, \mathbf{I} \cap (G^* - z))$-rigid, it is also $(G-u-z, \mathbf{I} \cap (G-u-z))$-rigid. This is a contradiction.
  - *Case* 2-2*:* $z \notin V(C)$.

    Since $z \notin V(C)$, we must have that $|N(G, z) \cap V(C)| \leq 1$. Hence, for any component H of $G^* - z$ containing w_p, regardless of whether $w_p \in V(C)$ (if $w_p \in V(C)$ then $w_1 \notin V(C)$ and vice versa), H is also a component of $G-z$, which means that t_p is $(G-z, \mathbf{I} \cap (G-z))$-rigid. Since w_p is $(G, \mathbf{I})$-movable and $z \notin V(C) \cap \mathbf{I}$, it follows by Lemma 3.4 that for every $j \in \{2, 3, \dots, p-1\}$, if $w_j \in \mathbf{I}$ then the token t_j on w_j is $(G-z, \mathbf{I} \cap (G-z))$-movable. As before, regardless of whether $w_j \in V(C)$ ($j \in \{2, 3, \dots, p-1\}$), the component H_j of $G-z$ containing w_j is also a component of $G^* - z$. Hence, for $j \in \{2, 3, \dots, p-1\}$, if $w_j \in \mathbf{I}$ then the token t_j on w_j is $(G^* - z, \mathbf{I} \cap (G^* - z))$-movable. Since t_p is $(G^*, \mathbf{I}^*)$-rigid, by Lemma 3.1, we must have that $w_1 \in \mathbf{I}$ and t_1 is $(G^* - z, \mathbf{I}^* \cap (G^* - z))$-rigid. Since t_p is also $(G^* - z, \mathbf{I}^* - z)$-rigid and $z \notin \mathbf{I}$, it follows from Lemma 3.4 that t_1 is $(G^*, \mathbf{I}^*)$-rigid, but this contradicts the assumption that w_p is the closest token to v that is $(G^*, \mathbf{I}^*)$-rigid.

□

Proof of Lemma 3.7.
The proof is similar to the proof of [3, Lemma 7].

The only-if-part is trivial. By Lemma 3.8, we know that Lemma 3.7 clearly holds for any cycle C.

Now, assume that the unicyclic graph G has at least one vertex of degree 1. We prove the if-part of the lemma by the induction on the number of tokens $|\mathbf{I}| = |\mathbf{J}|$. We choose an arbitrary safe degree-1 vertex v of G, whose unique neighbor is u. Since all tokens placed at vertices in $\mathbf{I}$ are $(G, \mathbf{I})$-movable, by Lemma 3.9 we can obtain an independent set $\mathbf{I}'$ of G such that $v \in \mathbf{I}'$ and $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{I}'$. By Lemma 3.10 all tokens in $\mathbf{I}' \setminus \{v\}$ are $(G^*, \mathbf{I}' \setminus \{v\})$-movable, where G^* is the subgraph defined in Lemma 3.10. Similarly, we can obtain an independent set $\mathbf{J}'$ of G such that $v \in \mathbf{J}'$, $\mathbf{J} \overset{G}{\leftrightsquigarrow} \mathbf{J}'$ and all tokens in $\mathbf{J}' \setminus \{v\}$ are $(G^*, \mathbf{J}' \setminus \{v\})$-movable. Apply the induction hypothesis to the pair of independent sets $\mathbf{I}' \setminus \{v\}$ and $\mathbf{J}' \setminus \{v\}$ of G^*. Then, we have $\mathbf{I}' \setminus \{v\} \overset{G^*}{\leftrightsquigarrow} \mathbf{J}' \setminus \{v\}$. Recall that both $u \notin \mathbf{I}'$ and $u \notin \mathbf{J}'$ hold, and u is the unique neighbor of v in G. Therefore, we can extend the reconfiguration sequence in G^* between $\mathbf{I}' \setminus \{v\}$ and $\mathbf{J}' \setminus \{v\}$ to a reconfiguration sequence in G between $\mathbf{I}$ and $\mathbf{J}$. We thus have $\mathbf{I} \overset{G}{\leftrightsquigarrow} \mathbf{J}$.

Put everything together, we finally have

Theorem 3.11. *The* SLIDING TOKEN *problem can be solved in $O(n^3)$ time for unicyclic graphs.*

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