Sliding tokens on unicyclic graphs

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The Reconfiguration Problem

- Instance:
  1. Collection of configurations.
  2. Allowed transformation rule(s).
- Question: For any two configurations A, B from the given collection, can A be transformed to B using the given rule(s)?

A classic example is the so-called 15-*puzzle*.

8	15	13	3
10		14	7
5	1	2	4
9	12	11	6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Jan van den Heuvel (2013). "The complexity of change". In: *Surveys in Combinatorics 2013*. Ed. by Simon R. Blackburn, Stefanie Gerke, and Mark Wildon. Cambridge University Press, pp. 127–160

Independent Set Reconfiguration (ISReconf)

- Instance:
  1. A graph $G = (V, E)$.

Independent Set Reconfiguration (ISRECONF)

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  2. Two independent sets I, J.

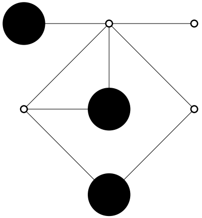

Figure: An independent set of a graph. Independent vertices are marked with black tokens.

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  3. "Reconfiguration" rules: TS, TJ, TAR.

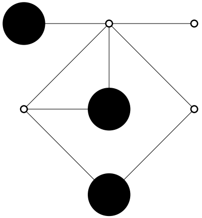
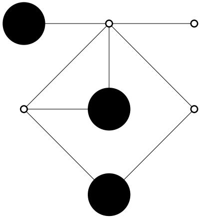


Figure: An independent set of a graph. Independent vertices are marked with black tokens.

1. TS: Slide tokens along edges (SLIDING TOKEN).
2. TJ: A token "jumps" from one vertex to another.
3. TAR: Add or Remove tokens.

Independent Set Reconfiguration (ISRECONF)

- INSTANCE:
  1. A graph $G = (V, E)$.
  2. Two independent sets I, J.
  3. "Reconfiguration" rules: TS, TJ, TAR.
- QUESTION: Can I be transformed to J using one of the given rules such that all intermediate sets are independent?



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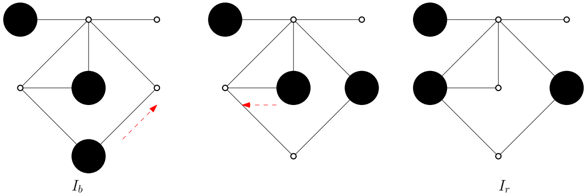
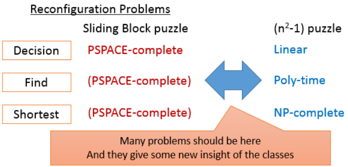


Figure: A YES-instance under TS rule.

Why study these problems?



(a) New insight of "complexity" inspired by games/puzzles (Picture © Ryuhei Uehara @ ICALP 2015).

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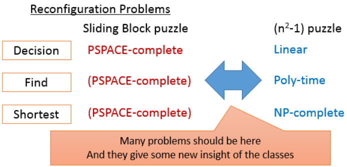

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CoRe 2015 Combinatorial Reconfiguration

(b) CoRe 2015 (Sendai, Japan) - Recent results + future directions of combinatorial reconfiguration.

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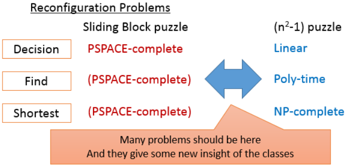

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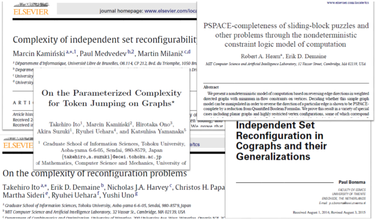

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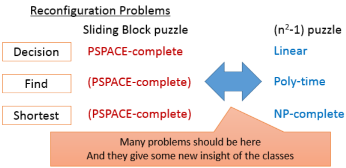

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2015 Combinatorial Reconfiguration

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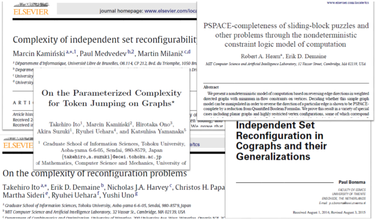

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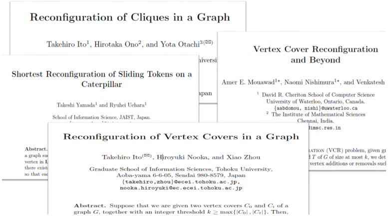

(d) Recently, several problems related to ISRECONF have been extensively studied.

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Linear-time algorithm for sliding tokens on trees

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- For bipartite graphs, all (G, I)-rigid tokens can be determined in linear time. (Fox-Epstein et al. 2015)
- What about graphs containing odd cycles?
  - Unicyclic graphs is a good start.

Rigid tokens in unicyclic graphs

A unicyclic graph is a connected graph that contains exactly one cycle.

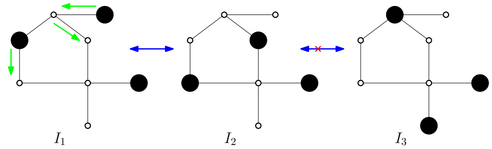

Figure: $I_1 \overset{G}{\leftrightsquigarrow} I_2$ and $I_2 \overset{G}{\not\leftrightsquigarrow} I_3$.

We say that the token t placed at $u \in I$ is (G, I)-rigid if for every independent set I' such that $I \overset{G}{\leftrightsquigarrow} I'$, $u \in I'$. Denote by $\mathsf{R}(G, I)$ the set of all (G, I)-rigid tokens.

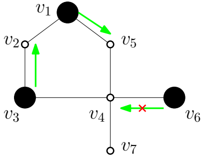

Figure: The token on v_6 is (G, I)-rigid, while the tokens on v_1 and v_3 are not.

Rigid tokens in unicyclic graphs

Lemma: Let I be an independent set of a unicyclic graph G ($|V(G)| = n$). Assume that the unique cycle C of G is of length k ($3 \leq k \leq |V(G)|$).

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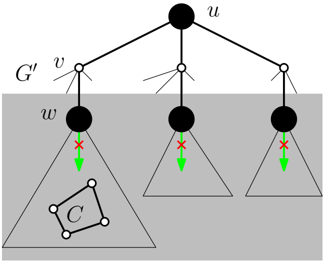
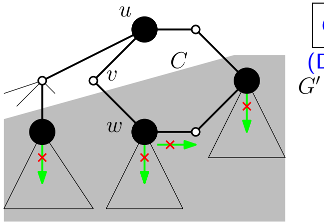
(i) The token t_w on w is $(G', I \cap G')$-rigid, where $G' = G - N[G, u]$.

(ii) $u \notin V(C)$, $\{v, w\} \subseteq V(C)$, the token t_w on w is not $(G', I \cap G')$-rigid, and for any independent set I' of G' such that $I \cap G' \overset{G'}{\leftrightsquigarrow} I'$, the path $P = C - v$ satisfies $|P \cap I'| = \lfloor k/2 \rfloor$.





$\boxed{O(n) \text{ for forests}}$

(Demaine et al. 2015)

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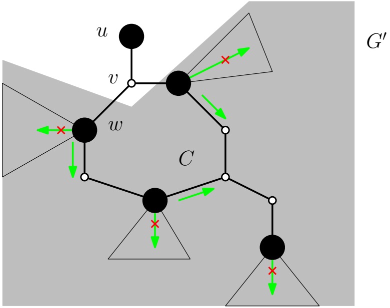

$\boxed{O(n^2)}$

Our contribution

(happen only when k is odd)

The following algorithm checks if $I \overset{G}{\leftrightsquigarrow} J$.

Step 1. Compute $\mathsf{R}(G, I)$ and $\mathsf{R}(G, J)$. If $\mathsf{R}(G, I) \neq \mathsf{R}(G, J)$, then return NO; otherwise go to **Step 2**.

Step 2. Delete the vertices in $\mathsf{R}(G, I) = \mathsf{R}(G, J)$ and its neighbors from G, and obtain a subgraph $\mathcal{F}$ consisting of q connected components $G_1, G_2, \ldots, G_q$. If the number of tokens in I and J are equal for every component, then return YES; otherwise, return NO.

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$\boxed{\text{Time: } O(n^2) \times (|I| + |J|) \Rightarrow O(n^3)}$

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$\boxed{\text{Time: } O(n)}$

Open Questions

1. Standard question: What is the complexity of ISRECONF for ... graph under ... rule?
   - bipartite graphs, cactus graphs, block graphs, interval graphs, etc.
   - TS, TJ, TAR.
2. Also, it is natural to ask questions about the length of the reconfiguration sequence (number of intermediate sets required to transform the source configuration to the target one).
   - The SLIDING TOKEN problem for trees was shown to be in P (Demaine et al. 2015), but the corresponding SHORTEST RECONFIGURATION problem is still open?
   - Known polynomial-time result:

     Takeshi Yamada and Ryuhei Uehara. "Shortest Reconfiguration of Sliding Tokens on a Caterpillar". In: *WALCOM 2016, Nepal, March 29-31, 2016 (To be appeared)*
3. The connection decision problem v.s. reconfiguration v.s. shortest reconfiguration provides a different view of "complexity" inspired by games/puzzles.