### 第157回アルゴリズム研究会



# Sliding tokens on unicyclic graphs

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# The Reconfiguration Problem



- Instance:
  - Collection of configurations.
  - 2 Allowed transformation rule(s).
- QUESTION: For any two configurations A, B from the given collection, can A be transformed to B using the given rule(s)?

A classic example is the so-called 15-puzzle.

8	15	13	3
10		14	7
5	1	2	4
9	12	11	6

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Jan van den Heuvel (2013). "The complexity of change". In: *Surveys in Combinatorics 2013*. Ed. by Simon R. Blackburn, Stefanie Gerke, and Mark Wildon. Cambridge University Press, pp. 127–160

# Independent Set Reconfiguration (ISRECONF)



- Instance:
  - **1** A graph G = (V, E).

# Independent Set Reconfiguration (ISReconf)



- Instance:

  - 2 Two independent sets I, J.

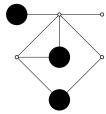


Figure: An independent set of a graph. Independent vertices are marked with black tokens.

# Independent Set Reconfiguration (ISReconf)



- Instance:

  - 2 Two independent sets I, J.
  - "Reconfiguration" rules: TS, TJ, TAR.

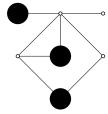


Figure: An independent set of a graph. Independent vertices are marked with black tokens.

- TS: Slide tokens along edges (SLIDING TOKEN).
- TJ: A token "jumps" from one vertex to another.
- TAR: Add or Remove tokens.

# Independent Set Reconfiguration (ISRECONF)



- Instance:

  - 2 Two independent sets I, J.
  - "Reconfiguration" rules: TS, TJ, TAR.
- QUESTION: Can I be transformed to J using one of the given rules such that all intermediate sets are independent?

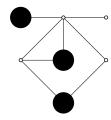


Figure: An independent set of a graph. Independent vertices are marked with black tokens.

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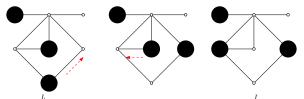
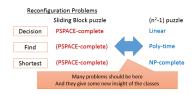


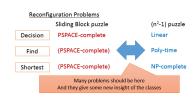
Figure: A YES-instance under TS rule.





(a) New insight of "complexity" inspired by games/puzzles (Picture © Ryuhei Uehara @ ICALP 2015).





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(b) CoRe 2015 (Sendai, Japan) - Recent results + future directions of combinatorial reconfiguration.







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(c) Several PSAPCE-hardness results were shown using reduction from ISRECONE.







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Shortest R	econfiguration of Sliding Tokens on a Caterpillar		Amer E.	Monawad <sup>1*</sup> , Naomi l	Nishimura <sup>1*</sup> , and Venkatesh
	Takeshi Yamada <sup>†</sup> and Ryuhei Uehara <sup>‡</sup>			David R. Cheriton School of Computer Science University of Waterloo, Ontario, Camada. (aabdunou, ninki)@univerloo.ca 7 The Institute of Mathematical Sciences	
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(c) Several PSAPCE-hardness results were shown using reduction from ISRECONE.

(d) Recently, several problems related to ISRECONF have been extensively studied.



Linear-time algorithm for sliding tokens on trees

Erik D. Demaine <sup>a</sup>, Martin L. Demaine <sup>a</sup>, Eli Fox-Epstein <sup>b</sup>, Duc A. Hoang <sup>c,\*</sup>, Takehiro Ito <sup>d,e</sup>, Hirotaka Ono <sup>f</sup>, Yota Otachi <sup>c</sup>, Ryuhei Uehara <sup>c</sup>, Takeshi Yamada <sup>c</sup>

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ABSTRACT

Suppose that we are given two independent sets  $I_b$  and  $I_T$  of a graph : and imagine that a token is placed on each vertex in  $I_b$ . Then, the SLIL is to determine whether there exists a sequence of independent set:

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#### Sliding Token on Bipartite Permutation Graphs

Eli Fox-Epstein<sup>1 (ES)</sup>, Duc A. Hoang<sup>2</sup>, Yota Otachi<sup>2</sup>, and Ryuhei Uehara<sup>2</sup>

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Abstract. Sliding Token is a natural reconfiguration problem in which vertices of independent sets are iteratively replaced by neighbors. We develop techniques that may be useful in answering the conjecture that SLIDING TOKEN is polynomial-time decidable on bipartite graphs. Along the way, we give efficient algorithms for Sliding Token on bipartite permutation and bipartite distance-hereditary graphs.

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- $\circ$  For bipartite graphs, all (G, I)-rigid tokens can be determined in linear time. (Fox-Epstein et al. 2015)
- What about graphs containing odd cycles?
  - Unicyclic graphs is a good start.



A unicyclic graph is a connected graph that contains exactly one cycle.

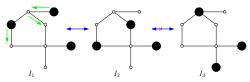


Figure:  $I_1 \overset{G}{\Leftrightarrow} I_2$  and  $I_2 \overset{G}{\Leftrightarrow} I_3$ .

We say that the token t placed at  $u \in I$  is (G,I)-rigid if for every independent set I' such that  $I \overset{G}{\Leftrightarrow} I'$ ,  $u \in I'$ . Denote by  $\mathsf{R}(G,I)$  the set of all (G,I)-rigid tokens.

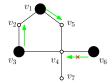


Figure: The token on  $v_6$  is (G, I)-rigid, while the tokens on  $v_1$  and  $v_3$  are not.



**Lemma**: Let I be an independent set of a unicyclic graph G (|V(G)| = n). Assume that the unique cycle C of G is of length k  $(3 \le k \le |V(G)|)$ .

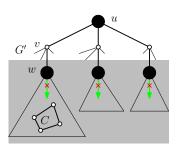


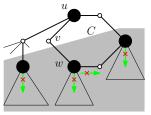
**Lemma**: Let I be an independent set of a unicyclic graph G (|V(G)| = n). Assume that the unique cycle C of G is of length k  $(3 \le k \le |V(G)|)$ . For any vertex  $u \in I$ , the token t on u is (G, I)-rigid if and only if for every vertex  $v \in N(G, u)$ , there exists a vertex  $v \in (N(G, v) \setminus \{u\}) \cap I$  satisfying one of the following conditions:



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- (i) The token  $t_w$  on w is  $(G', I \cap G')$ -rigid, where G' = G N[G, u].
- (ii)  $u \notin V(C)$ ,  $\{v, w\} \subseteq V(C)$ , the token  $t_w$  on w is not  $(G', I \cap G')$ -rigid, and for any independent set I' of G' such that  $I \cap G' \overset{G'}{\Leftrightarrow} I'$ , the path P = C v satisfies  $|P \cap I'| = \lfloor k/2 \rfloor$ .





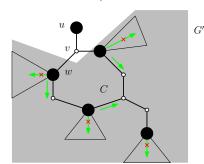
O(n) for forests

(Demaine et al. 2015



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 $O(n^2)$ 

Our contribution (happen only when k is odd)



The following algorithm checks if  $I \stackrel{G}{\Leftrightarrow} J$ .

- **Step 1.** Compute R(G,I) and R(G,J). If  $R(G,I) \neq R(G,J)$ , then return NO; otherwise go to **Step 2**.
- **Step 2.** Delete the vertices in R(G,I) = R(G,J) and its neighbors from G, and obtain a subgraph  $\mathcal F$  consisting of q connected components  $G_1,G_2,\ldots,G_q$ . If the number of tokens in I and J are equal for every component, then return YES; otherwise, return NO.



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Time:  $O(n^2) \times (|I| + |J|) \Rightarrow O(n^3)$ 

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Time: O(n)

### **Open Questions**



- Standard question: What is the complexity of ISRECONF for ... graph under ... rule?
  - o bipartite graphs, cactus graphs, block graphs, interval graphs, etc.
  - o TS, TJ, TAR.
- Also, it is natural to ask questions about the length of the reconfiguration sequence (number of intermediate sets required to transform the source configuration to the target one).
  - The SLIDING TOKEN problem for trees was shown to be in P (Demaine et al. 2015), but the corresponding SHORTEST RECONFIGURATION problem is still open?
  - Known polynomial-time result:
     Takeshi Yamada and Ryuhei Uehara. "Shortest Reconfiguration of Sliding Tokens on a Caterpillar". In: WALCOM 2016, Nepal, March 29-31, 2016 (To be appeared)
- The connection decision problem v.s. reconfiguration v.s. shortest reconfiguration provides a different view of "complexity" inspired by games/puzzles.