A study on the structure of reconfiguration graphs and related problems

Summary of my research at VIASM in 2024

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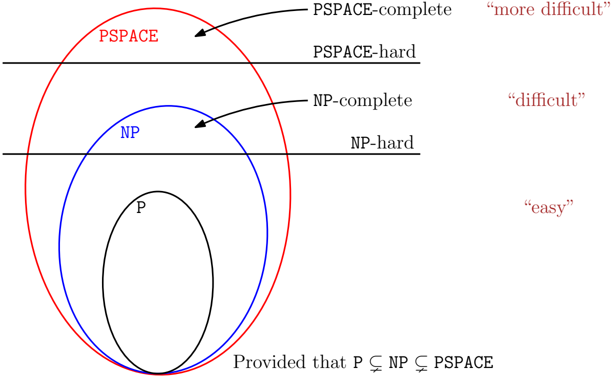
Seminar at VIASM

Outline

Introduction

Complexity Classes: P, NP, and PSPACE

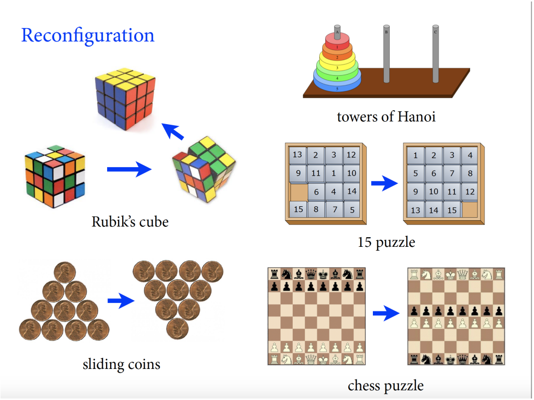
- We focus on *decision problems* (output YES or NO).
- Complexity Classes:
  - P: Problems solvable in polynomial time.
  - NP: Problems verifiable in polynomial time.
  - PSPACE: Problems solvable in polynomial space.



What is Combinatorial Reconfiguration?

Reconfiguration Setting

- A description of what *states* (≡ *configurations*) are
- One or more *allowed moves* between states (≡ *reconfiguration rule(s)*)



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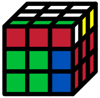
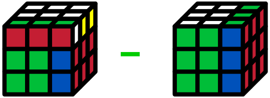
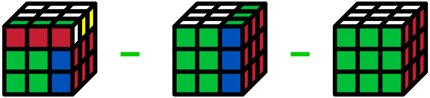
What is Combinatorial Reconfiguration?

Two major viewpoints: as a *process* or as a *graph*

Reconf. Process	**Reconf. Graph**	**Example** RUBIK'S CUBE
state	node	
move *[reconf. rule]*	edge	
seq. of moves *[reconf. sequence]*	path	

What is Combinatorial Reconfiguration?

Example 1 (Token Reconfiguration)

- **States:**
  - Each vertex of a graph G contains at most one token.
  - Each state is a set of tokens satisfying a property $\mathcal{P}$ (e.g., independent sets, dominating sets).

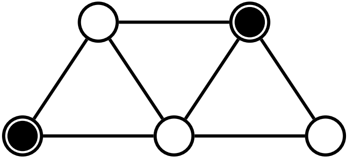


Figure: Independent Set

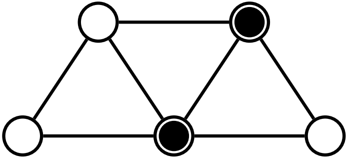


Figure: Dominating Set

- **Moves:**
  - TS: A token moves to an adjacent unoccupied vertex.
  - TJ: A token moves to any unoccupied vertex.
  - TAR(k): A token is added or removed, keeping the token-set size within bounds.
  - Each move must *preserve property $\mathcal{P}$*.

What is Combinatorial Reconfiguration?

Example 2 (Vertex-Coloring Reconfiguration)

- **States:**
  - Each vertex of a graph G is colored with a color from a given set of $k \geq 1$ colors such that no two adjacent vertices share the same color. This is known as a *(proper) k-coloring* of G.
  - Each state is a (proper) k-coloring of G.

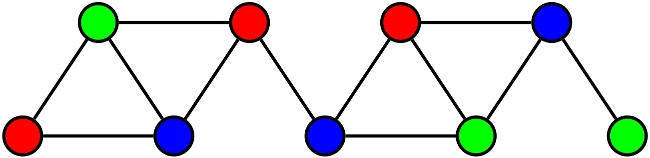


Figure: A proper 3-coloring of a graph G

- **Moves:**
  - **Recoloring:** A vertex can be recolored with a different color
  - Again, each move must *preserve the property of being a (proper) k-coloring*

What is Combinatorial Reconfiguration?

Two major directions: *Algorithmic* and *Graph-Theoretic*

- **Algorithmic Questions**
  - REACHABILITY: Given two states S and T, is there a sequence of moves that *transforms S into T*?
  - SHORTEST TRANSFORMATION: Given two states S and T and some positive integer ℓ, is there a sequence of moves that *transforms S into T using at most ℓ moves*?
  - CONNECTIVITY: Is there a sequence of moves *between any pair of states*?
  - and so on
- **Graph-Theoretic Questions**
  - GRAPH PROPERTIES: Is the reconfiguration graph *connected*? *bipartite*? *Eulerian*? *Hamiltonian*?, and so on
  - GRAPH CLASSIFICATION: Does the reconfiguration graph *belong to some specific graph class* (e.g., planar graphs, perfect graphs, etc.)?
  - and so on

Surveys and Wiki Page

- **General Surveys**
  - Jan van den Heuvel (2013). "The Complexity of Change". In: *Surveys in Combinatorics*. Vol. 409. London Mathematical Society Lecture Note Series. Cambridge University Press, pp. 127–160. DOI: 10.1017/cbo9781139506748.005
  - Naomi Nishimura (2018). "Introduction to Reconfiguration". In: *Algorithms* 11.4, p. 52. DOI: 10.3390/a11040052
- **Surveys on Specific Problems**
  - C.M. Mynhardt and S. Nasserasr (2019). "Reconfiguration of Colourings and Dominating Sets in Graphs". In: *50 years of Combinatorics, Graph Theory, and Computing*. Ed. by Fan Chung et al. 1st. CRC Press, pp. 171–191. DOI: 10.1201/9780429280092-10
  - Nicolas Bousquet, Amer E. Mouawad, Naomi Nishimura, and Sebastian Siebertz (2024). "A survey on the parameterized complexity of reconfiguration problems". In: *Computer Science Review* 53. (article 100663). DOI: 10.1016/j.cosrev.2024.100663
- **Online Wiki:** `http://reconf.wikidot.com/` (I am one of the maintainers of this site)

Research Summary

Research Summary

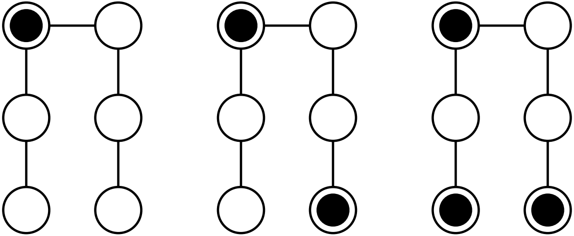
Research Questions

1. How does the complexity change when adding a "distance constraint"?
2. How does the complexity change when adding a "directed constraint"?
3. Which graphs are reconfiguration graphs?

Research Summary

Question 1: How does the complexity change when adding a "distance constraint"?

- Let r be a positive integer and G be a graph
- A vertex v of G *r-dominates* every vertex within distance r from it
- A *distance-r dominating set (DrDS)* of a graph G is a vertex subset $D \subseteq V(G)$ such that every vertex of G is r-dominated by some vertex in D [Generalization of "dominating set"]



D1DS: NO
D2DS: NO

D1DS: NO
D2DS: YES

D1DS: YES
D2DS: YES

Research Summary

Question 1: How does the complexity change when adding a "distance constraint"?

Distance-r Dominating Set Reconfiguration (DrDSR)

- **Input:** Two distance-r dominating sets S and T of a graph G, and a reconfiguration rule $\mathsf{R} \in \{\mathsf{TS}, \mathsf{TJ}\}$.
- **Question:** Is there an R-sequence between S and T?

- The problem is well-studied for $r = 1$ (i.e., Dominating Set Reconfiguration).
- We extend the complexity analysis to $r \geq 2$.

Research Summary

Question 1: How does the complexity change when adding a "distance constraint"?

Main Result 1 [Banerjee and **Hoang** 2024]

DrDSR ($r \geq 1$) is PSPACE-complete on planar graphs of maximum degree 3 and bounded bandwidth.

- Previous results for $r = 1$ were limited to graphs with maximum degree 6.
- We extend these results to $r \geq 2$ and improve the bounds.

Main Result 2 [Banerjee and **Hoang** 2024]

DrDSR ($r \geq 1$) on split graphs: PSPACE-complete when $r = 1$ (which is already known) but in P when $r \geq 2$ (which we prove).

- An interesting complexity dichotomy.
- We establish bounds on the shortest reconfiguration sequence length for $r = 2$. (The case $r \geq 3$ is straightforward.)

Research Summary

Question 1: How does the complexity change when adding a "distance constraint"?

- Let $d \geq 1$ and $k \geq d+1$ be integers and G be a graph
- A *(d, k)-coloring* of G is a vertex-coloring of G using k given colors such that no two vertices within distance d share the same color [Generalization of "(proper) k-coloring"]

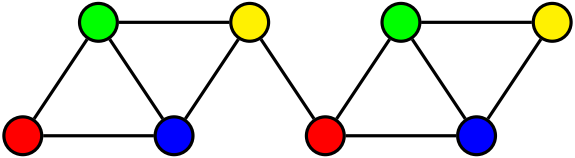


Figure: An example of a $(2, 4)$-coloring of a graph which is also a $(1, 4)$-coloring (i.e., a (proper) 4-coloring) but not a $(3, 4)$-coloring

Research Summary

Question 1: How does the complexity change when adding a "distance constraint"?

(d, k)-Coloring Reconfiguration ((d, k)-CR)

- **Input:** Two (d, k)-colorings α and β of a graph G, with one vertex recolored per step.
- **Question:** Is there a recoloring sequence from α to β or vice versa?

- Generalizes Vertex-Coloring Reconfiguration $(d = 1)$
- Analyzes complexity for $d \geq 2$

Research Summary

Question 1: How does the complexity change when adding a "distance constraint"?

Main Result 1 [Banerjee, Engels, and **Hoang** 2024b]

(d, k)-CR is PSPACE-complete on graphs that are planar, bipartite, and 2-degenerate for $d \geq 2$ and $k = \Omega(d^2)$

- The problem remains challenging for very restricted graph classes.
- Previous hardness results are limited to $d = 1$ and $k \geq 4$ on planar or bipartite graphs.

Main Result 2 [Banerjee, Engels, and **Hoang** 2024b]

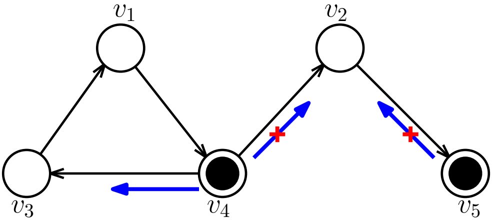
(d, k)-CR on split graphs: PSPACE-complete for $d = 2$ and large k, but in P for $d \geq 3$ and any $k \geq d + 1$

- An interesting complexity dichotomy
- For $d = 1$, the problem is known to be in P

Research Summary

Question 2: How does the complexity change when adding a "directed constraint"?

- Let G be an *oriented graph* (i.e., each edge has a unique direction). The *underlying undirected graph* G^{und} is obtained by ignoring edge directions.
- A set S is *independent* in an undirected graph if no two members are joined by an edge. In an oriented graph, S is independent if it is independent in G^{und}.
- **Directed Token Sliding (**DTS**):** A token can only move along a directed edge.



Research Summary

Question 2: How does the complexity change when adding a "directed constraint"?

Directed Token Sliding (DTS)

- **Input:** Two independent sets S and T on an oriented graph, and the reconfiguration rule DTS.
- **Question:** Is there a DTS-sequence from S to T or vice versa?

- Generalizes Independent Set Reconfiguration under TS on undirected graphs
- Introduced in [Ito et al. 2022]
- We study the problem's complexity on various oriented graph classes

Research Summary

Question 2: How does the complexity change when adding a "directed constraint"?

Main Result [Banerjee, Engels, and **Hoang** 2024a]

DTS is PSPACE-complete on:

- planar graphs
- bipartite graphs
- bounded treewidth graphs
- split graphs

- Previous results cover general oriented graphs (PSPACE-complete), directed acyclic graphs (NP-complete), and oriented trees (in P) [Ito et al. 2022].

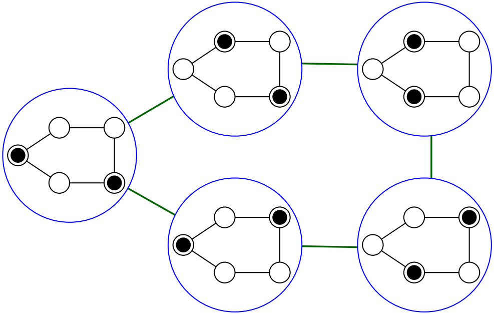
Research Summary

Question 3: Which graphs are reconfiguration graphs?

Reconfiguration graphs of Independent Set under TS

- Graph $\mathsf{TS}_k(G)$
  - **Vertices:** Size-k independent sets of G
  - **Edges:** Two sets are adjacent if one can be transformed into the other by a single TS-move

- The structure and properties of $\mathsf{TS}_k(G)$ have not been well-understood.
- Introduced in [Avis and **Hoang** 2023].



Research Summary

Question 3: Which graphs are reconfiguration graphs?

Main Result [Avis and **Hoang** 2024]

Necessary and sufficient conditions for $\mathsf{TS}_k(G)$ to be a tree/forest:

- The problem remains open for acyclic G
- We provide forbidden-subgraph characterizations for acyclic G with $k = 2, 3$, and conjecture for $k \geq 4$

Necessary and sufficient conditions for a tree/forest to be a TS_k-graph:

- For every k-ary tree T, there is a graph G such that $\mathsf{TS}_{k+1}(G) \simeq T$
- More results are in [Avis and **Hoang** 2024]

Papers that acknowledge the support of VIASM

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Papers **partially completed** in 2024 at VIASM

1. David Avis and **Duc A. Hoang** (2024). "A Note on Acyclic Token Sliding Reconfiguration Graphs of Independent Sets". In: *Ars Combinatoria* 159, pp. 133–154. DOI: 10.61091/ars159-12
2. Niranka Banerjee and **Duc A. Hoang** (2024). "The Complexity of Distance-r Dominating Set Reconfiguration". In: *Proceedings of COCOON 2024*. LNCS. arXiv: 2310.00241
   - Submitted to a journal
   - **To appear in COCOON 2024 proceedings, published by Springer in January 2025**
3. Niranka Banerjee, Christian Engels, and **Duc A. Hoang** (2024b). "Distance Recoloring". In: *arXiv preprint*. arXiv: 2402.12705
   - Submitted to a conference

Papers that acknowledge the support of VIASM

Papers **primarily completed** in 2024 at VIASM

1. Quan N. Lam, Huu-An Phan, and **Duc A. Hoang** (2024). "A Note on Reconfiguration Graphs of Cliques". Unpublished
   - Submitted to a journal
   - **Vietnam-Polymath-REU 2023-2024** (Quan from VNU-HCMC and An from NTU Singapore. An contributed significantly to this work)
2. Niranka Banerjee, Christian Engels, and **Duc A. Hoang** (2024a). "Directed Token Sliding". In: *arXiv preprint*. arXiv: 2411.16149
   - Submitted to a conference
   - **Work began in May 2024 during Christian and Niranka's visit to VIASM**
3. Meike Hatzel, **Duc A. Hoang**, and Marcelo Garlet Milani (2024). "Some results on Token Sliding graphs". Unpublished
   - Work in Progress (Title tentative, arXiv version forthcoming.)
   - **Started in February 2024 during the WOGGA3 workshop in Okinawa, Japan.** (WOGGA4 is taking place this week (Dec. 16–19, 2024) in Kyoto, Japan. I attended some online sessions.)

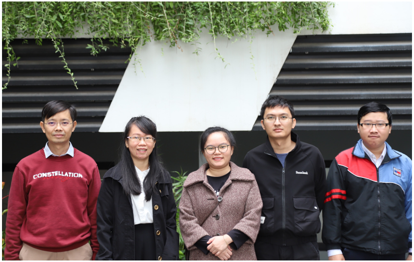
Future Plans

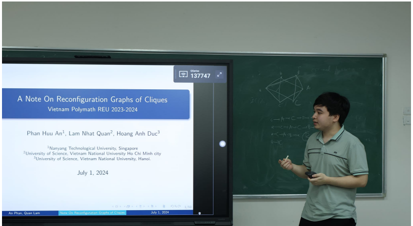
Future Plans

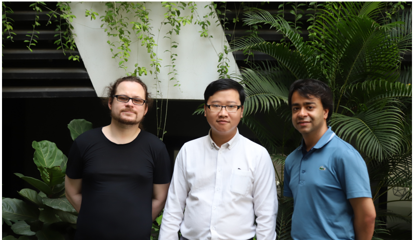
- **Research:** Continue investigating reconfiguration problems from both algorithmic and graph-theoretic perspectives.
- **Research Activities**
  - **Collaborations and Mentorship:** Vietnam-Polymath-REU 2024-2025, Undergraduate Thesis at VNU-HUS (work in progress).
  - **Conference/Workshop:** The 3rd Vietnam-Korea Joint Workshop on Selected Topics in Mathematics (upcoming, contributed talks in "Combinatorics and Discrete Mathematics" session).

Acknowledgments: I would like to express my gratitude to

- **Vietnam Institute for Advanced Study in Mathematics (VIASM)** for providing an excellent research environment and working condition.
- **My collaborators and co-authors** for their valuable contributions to our joint research projects.







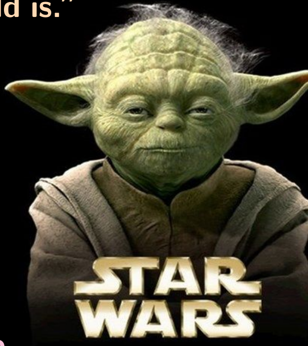
"Pass on what you have learned. Strength. Mastery. But weakness, folly, failure also. Yes, failure most of all."

"A way, there always is."

"May the Force be with you."

"Truly wonderful, the mind of a child is."

Thanks For Your Attention!



"In a dark place we find ourselves, and a little more knowledge lights our way."

"Always two there are. No more, no less. A master and an apprentice."

"Do or do not. There is no try."

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Lam, Quan N., Huu-An Phan, and **Duc A. Hoang** (2024). "A Note on Reconfiguration Graphs of Cliques". Unpublished.

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