A note regarding "Sliding Tokens on Block Graphs"

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1 Introduction

In this note, we introduce a counter-example for Proposition 6 of [2] and the progress of resolving this issue. This example was provided by Mariana Teatini Ribeiro and Vinícius Fernandes dos Santos. This issue has also been announced in [3]. (Duc A. Hoang (me) and Ryuhei Uehara are co-authors in both publications.)

2 The problem

Let I, J be two given independent sets of a graph G. Imagine that the vertices of an independent set are viewed as tokens (coins). A token is allowed to move (or slide) from one vertex to one of its neighbors. The SLIDING TOKEN problem asks whether there exists a sequence of independent sets of G starting from I and ending with J such that each intermediate member of the sequence is obtained from the previous one by moving a token according to the allowed rule. If such a sequence exists, we write $I \overset{G}{\leftrightsquigarrow} J$. In [2], we claimed that this problem is solvable in polynomial time when the input graph is a block graph—a graph whose blocks (i.e., maximal 2-connected subgraphs) are cliques.

3 Proposition 6 and its counter-example

Let I be an independent set of a graph G. Let $W \subseteq V(G)$ and assume that $I \cap W \neq \emptyset$. We say that a token t placed at some vertex $u \in I \cap W$ is (G, I, W)-*confined* if for every J such that $I \overset{G}{\leftrightsquigarrow} J$, t is always placed at some vertex of W. In other words, t can only be slid along edges of $G[W]$. Let H be an induced subgraph of G. H is called (G, I)-*confined* if $I \cap V(H)$ is a maximum independent set of H and all tokens in $I \cap V(H)$ are $(G, I, V(H))$-confined.

Mariana Teatini Ribeiro and Vinícius Fernandes dos Santos showed us a counter-example of the following proposition

Proposition 1 ([2, Proposition 6]). *Let I be an independent set of a block graph G. Let $w \in V(G)$. Assume that no block of G containing w is (G, I)-confined.*

If there exists some vertex $x \in N_G[w] \cap I$ such that the token t_x placed at x is $(G, I, N_G[w])$-confined, then x is unique. Consequently, there must be some independent set J such that $I \overset{G}{\leftrightsquigarrow} J$ and $N_G[w] \cap J = \{x\}$. Moreover, let H be the graph obtained from G by turning $N_G[w]$ into a clique, called B_w. Then t_x is $(G, J, N_G[w])$-confined if and only if B_w is (H, J)-confined.

The statement **Moreover, let H be the graph obtained from G by turning $N_G[w]$ into a clique, called B_w. Then t_x is $(G, J, N_G[w])$-confined if and only if B_w is (H, J)-confined** is indeed not correct. Figure 1 illustrates

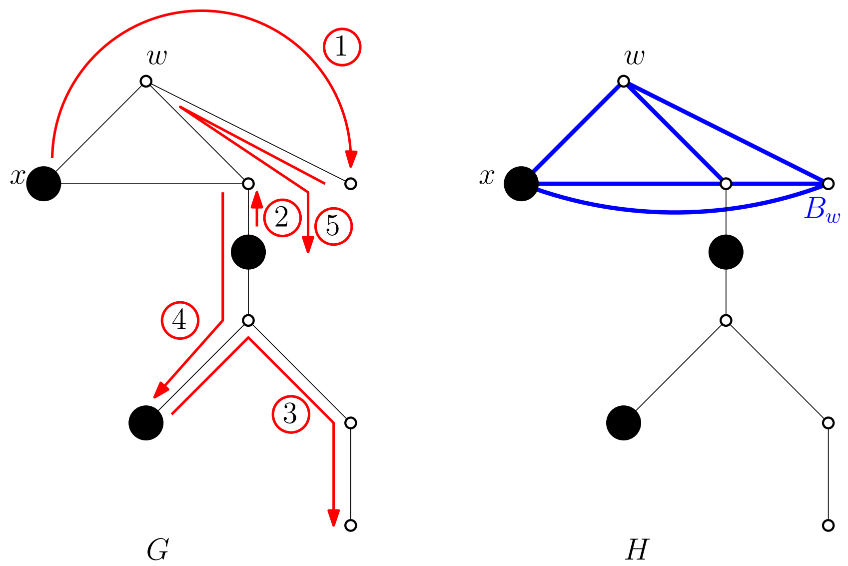

Figure 1: A counter-example of Proposition 1.

a counter-example of this statement. Here, the block B_w of H (containing t_x) is (H, J)-confined but t_x is not $(G, J, N_G[w])$-confined. The red arrows in Figure 1 describes a way of moving t_x out of $N_G[w]$. (The numbers inside red circles indicate the order of performing the steps described by the red arrows.)

4 Progress on resolving the issue

Recently, a polynomial-time algorithm for solving Sliding Token on block graphs was proposed by Francis and Prabhakaran [1].

References

[1] Mathew C. Francis and Veena Prabhakaran. Token sliding independent set reconfiguration on block graphs. *arXiv preprint, arXiv:2410.07060*, 2024. URL `https://arxiv.org/abs/2410.07060`.

[2] Duc A. Hoang, Eli Fox-Epstein, and Ryuhei Uehara. Sliding Tokens on Block Graphs. In *Proceedings of WALCOM 2017*, volume 10167 of *LNCS*, pages 460–471. Springer, 2017. doi: 10.1007/978-3-319-53925-6_36.

[3] Duc A. Hoang, Amanj Khorramian, and Ryuhei Uehara. Shortest Reconfiguration Sequence for Sliding Tokens on Spiders. In Pinar Heggernes, editor, *Proceedings of CIAC 2019*, volume 11485 of *LNCS*, pages 262–273. Springer, 2019. doi: 10.1007/978-3-030-17402-6_22.