Sliding tokens on block graphs

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Outline

1. Reconfiguration problems and moving tokens on graphs
2. Sliding tokens on block graphs in polynomial time
3. Open questions

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General framework: Reconfiguration Problems

- INSTANCE:
  1. Collection of configurations.
  2. Allowed transformation rule(s).
- QUESTION: Decide if configuration A can be transformed to configuration B using the given rule(s), while maintaining a configuration throughout.

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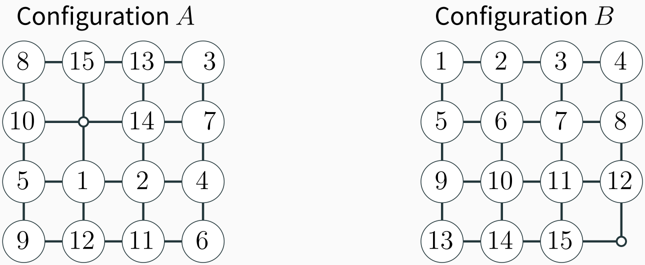

Figure 1: The 15-puzzles.

General framework: Reconfiguration Problems

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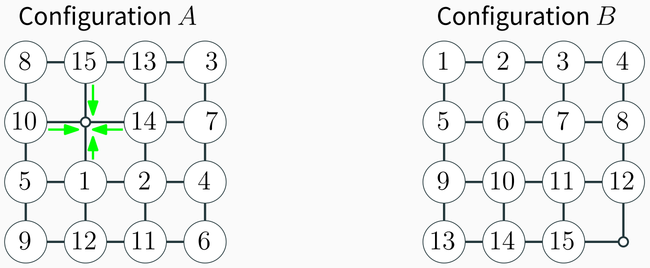

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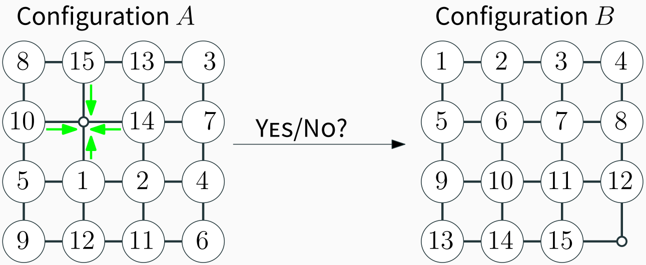

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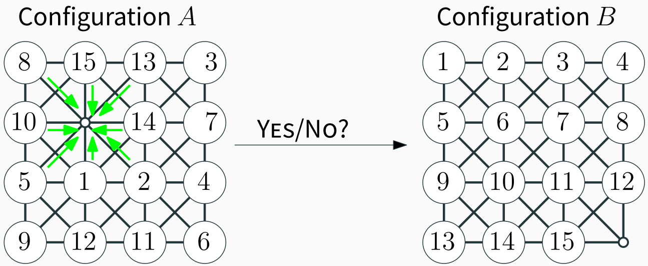
Generalization of 15-puzzles

- **Graphs:** grid, trees, block, planar, perfect, etc.
- **Rules:** Token Sliding, Token Jumping, Token Swapping, etc.
- **Labels:** distinct labels for all tokens, some tokens can be of the same label, no label, etc.
- **Restrictions:** no restriction, independent set, dominating set, etc.



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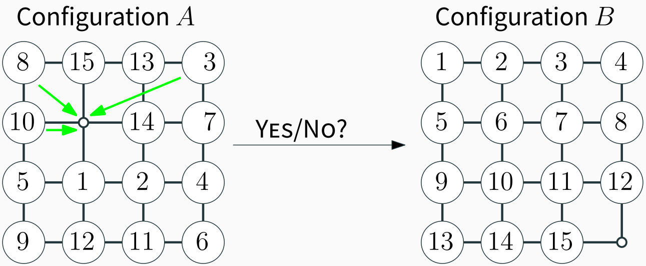

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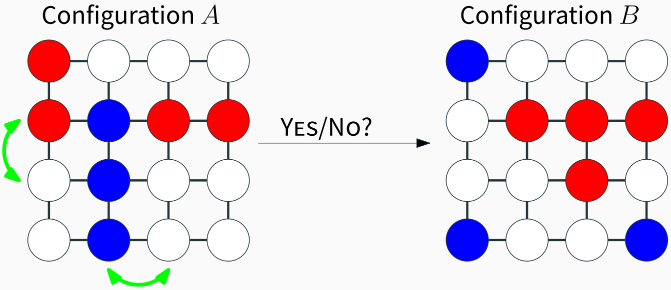

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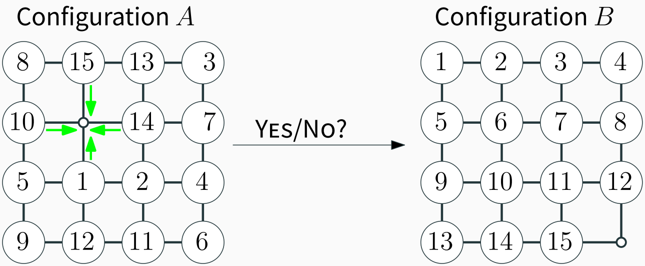

Our Problem: Sliding Token for block graphs

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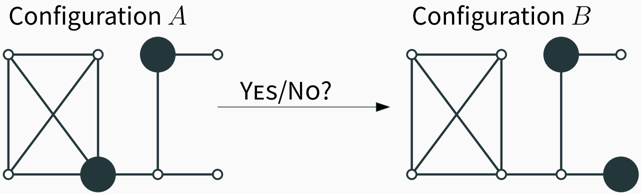
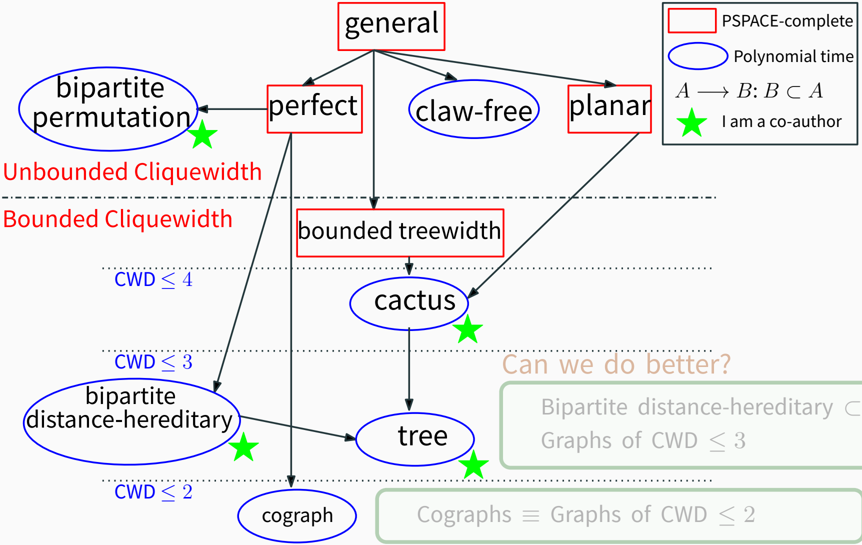


Block graphs: Every block (i.e., maximal 2-connected subgraph) is a clique.

SLIDING TOKEN - **Complexity Status**



Can we do better?

Bipartite distance-hereditary $\subset$ Graphs of CWD ≤ 3

Cographs $\equiv$ Graphs of CWD ≤ 2

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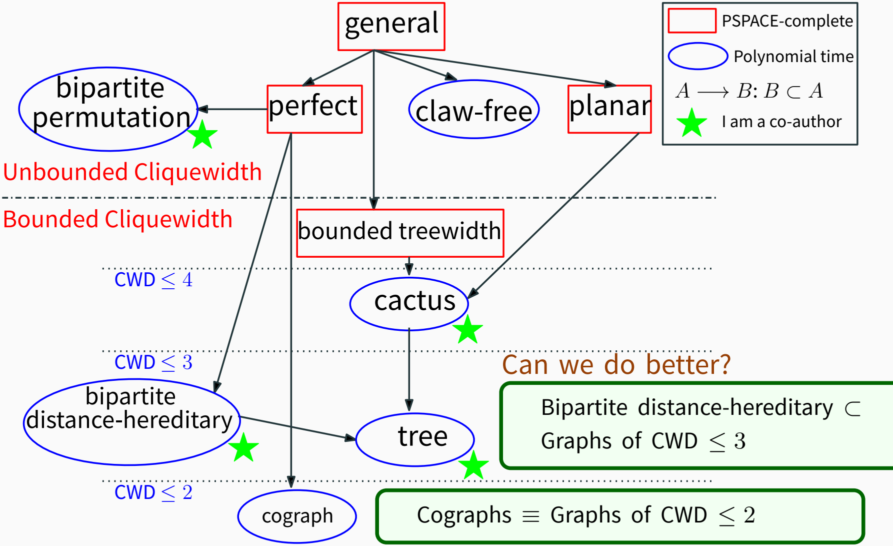

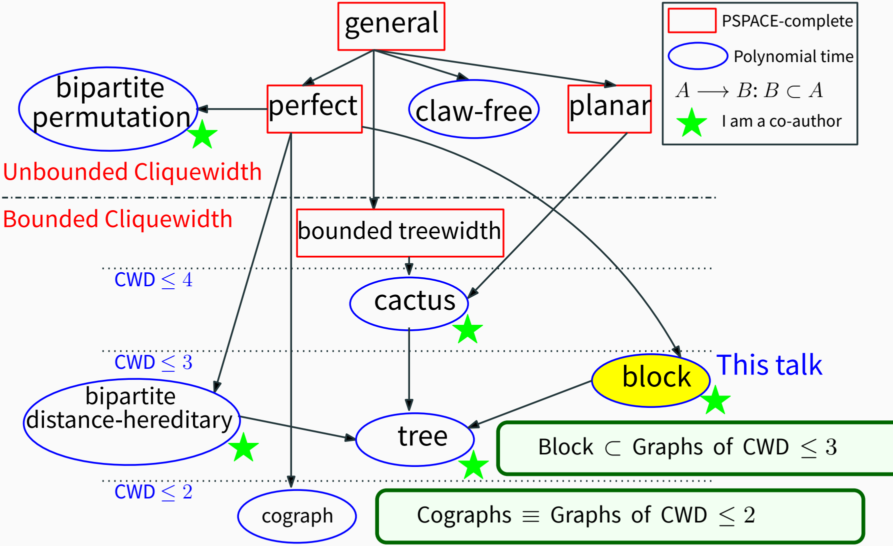
general
bipartite permutation
perfect
claw-free
planar
PSPACE-complete
Polynomial time
$A \longrightarrow B$: $B \subset A$
I am a co-author
Unbounded Cliquewidth
Bounded Cliquewidth
bounded treewidth
CWD ≤ 4
cactus
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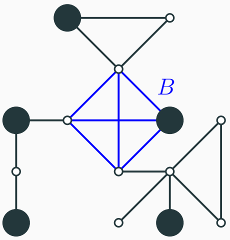
Key structure: (G, I)-confined clique

(G, I)-confined clique: The "inside" token cannot be slid "out."



Lemma 1: One can find all (G, I)-confined cliques in time $O(m^2)$, where $m = |E(G)|$.

Lemma 2: For two independent sets I, J, if the set of confined cliques for I and J are different, then I cannot be reconfigured to J (and vice versa).

Lemma 3: If there are no confined cliques for both I and J, then I can be reconfigured to J iff $|I| = |J|$.

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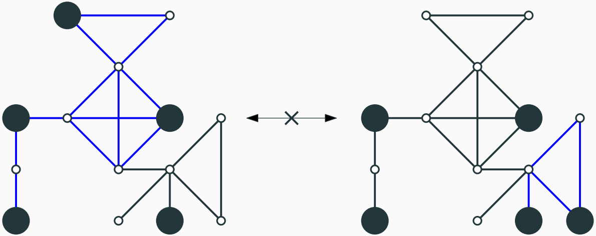

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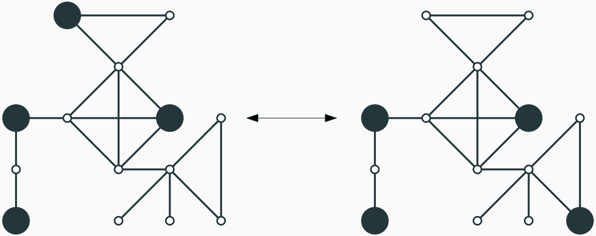

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Our Algorithm

Given an instance (G, I, J) of Sliding Token, where I, J are two independent sets of a block graph G.

1. Find all confined cliques for both I and J. If the set of confined cliques for I and J are different, return NO.
   Otherwise, remove all confined cliques for I and J (they are the same). Let G' be the resulting graph.
2. For each component F of G', if $|I \cap F| \neq |J \cap F|$, return NO.
   Otherwise, return YES.

Running time: $O(m^2 + n)$, where $m = |E(G)|$ and $n = |V(G)|$.

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- Whether one can solve SLIDING TOKEN for block graphs in linear time.
- When considering graphs of cliquewidth at most 3, distance-hereditary graphs is more general than block graphs. SLIDING TOKEN remains open for distance-hereditary graphs.

SLIDING TOKEN is also polynomial-time solvable for bipartite distance-hereditary graphs [Fox-Epstein, Hoang, Otachi, and Uehara 2015] and cographs [Kamiński, Medvedev, and Milanič 2012].

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Appendix

- Recent results on studying ISReconf
- Cliquewidth

Recent results on studying ISRECONF

Graph	Rule(s)	Complexity	Paper(s)
planar	TS, TJ, TAR	PSPACE-complete	Hearn and Demaine 2005
general	TS, TJ, TAR	PSPACE-complete	Ito et al. 2011
line	TJ, TAR	P	
perfect	TS, TJ, TAR	PSPACE-complete	Kamiński, Medvedev, and Milanič 2012
even-hole-free	TJ, TAR	P	
cograph (P_4-free)	TS	P	
cograph (P_4-free)	TJ, TAR	P	Bonsma 2016
bounded bandwidth	TS, TJ, TAR	PSPACE-complete	Wrochna 2014
claw-free	TS, TJ	P	Bonsma, Kamiński, and Wrochna 2014
tree	TS	P	Demaine et al. 2015
bipartite permutation	TS	P	Fox-Epstein, Hoang, Otachi, and Uehara 2015
bipartite distance-hereditary	TS	P	
cactus	TS	P	Hoang and Uehara 2016
block	TS	P	Hoang, Fox-Epstein, and Uehara 2017

Table 1: Recent results on studying ISRECONF under Token Sliding (TS), Token Jumping (TJ), and Token Addition and Removal (TAR).

Cliquewidth I

The *cliquewidth* of a graph G, denoted by $cwd(G)$, is the minimum number of labels needed to construct G using the following four operations:

1. Creation of a new vertex v with label i (denoted by $i(v)$).
2. Disjoint union of two labelled graphs G and H (denoted by $G \oplus H$).
3. Joining by an edge each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).
4. Renaming label i to j (denoted by $\rho_{i \to j}$)

Cliquewidth II

Every graph can be defined by an algebraic expression using these four operations. For instance, a chordless path on five consecutive vertices a, b, c, d, e can be defined as follows:

$$\eta_{2,3}(\rho_{3\to 1}(\eta_{2,3}(\rho_{2\to 1}(\eta_{2,3}(\eta_{1,2}(1(a)\oplus 2(b))\oplus 3(c)))\oplus 2(d)))\oplus 3(e))$$

Such an expression is called a k-*expression* if it uses at most k different labels. Thus the cliquewidth of G is the minimum k for which there exists a k-expression defining G. For instance, from the above example we conclude that $cwd(P_5) \leq 3$.

Cliquewidth of some well-known graphs

- Cographs (graphs having no P_4 as induced subgraph) are exactly the graphs of cliquewidth at most 2.
- A complete graph K_n is of cliquewidth at most 2.
- A tree (and hence a forest) is of cliquewidth at most 3.

Theorem (González-Ruiz, Marcial-Romero, and Hernández-Servín 2016)

The cliquewidth of a cactus is at most 4.

Theorem (Golumbic and Rotics 2000)

The cliquewidth of a distance-hereditary graph is at most 3. Consequently, any subclass of distance-hereditary graphs is of cliquewidth at most 3.

Bibliography I

Bonsma, Paul (2016). "Independent Set Reconfiguration in Cographs and their Generalizations". In: *Journal of Graph Theory* 83.2, pp. 164–195. DOI: 10.1002/jgt.21992.

Bonsma, Paul, Marcin Kamiński, and Marcin Wrochna (2014). "Reconfiguring Independent Sets in Claw-Free Graphs". In: *Proceedings of SWAT 2014*. Ed. by R. Ravi and IngeLi Gørtz. Vol. 8503. LNCS. Springer, pp. 86–97. DOI: 10.1007/978-3-319-08404-6_8.

Demaine, Erik D., Martin L. Demaine, Eli Fox-Epstein, Duc A. Hoang, Takehiro Ito, Hirotaka Ono, Yota Otachi, Ryuhei Uehara, and Takeshi Yamada (2015). "Linear-time algorithm for sliding tokens on trees". In: *Theoretical Computer Science* 600, pp. 132–142. DOI: 10.1016/j.tcs.2015.07.037.

Bibliography II

Fox-Epstein, Eli, Duc A. Hoang, Yota Otachi, and Ryuhei Uehara (2015). "Sliding Token on Bipartite Permutation Graphs". In: *Proceedings of ISAAC 2015*. Ed. by Khaled Elbassioni and Kazuhisa Makino. Vol. 9472. LNCS. Springer, pp. 237–247. DOI: 10.1007/978-3-662-48971-0_21.

Golumbic, Martin Charles and Udi Rotics (2000). "On the clique-width of some perfect graph classes". In: *International Journal of Foundations of Computer Science* 11.03, pp. 423–443. DOI: 10.1142/S0129054100000260.

González-Ruiz, J. Leonardo, J. Raymundo Marcial-Romero, and J.A. Hernández-Servín (2016). "Computing the Clique-width of Cactus Graphs". In: *Electronic Notes in Theoretical Computer Science* 328, pp. 47–57. DOI: 10.1016/j.entcs.2016.11.005.

Hearn, Robert A. and Erik D. Demaine (2005). "PSPACE-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation". In: *Theoretical Computer Science* 343.1, pp. 72–96. DOI: 10.1016/j.tcs.2005.05.008.

Bibliography III

Hoang, Duc A., Eli Fox-Epstein, and Ryuhei Uehara (2017). "Sliding token on block graphs". In: *Proceedings of WALCOM 2017*. Ed. by Sheung-Hung Poon, Md. Saidur Rahman, and Hsu-Chun Yen. Vol. 10167. LNCS. Springer, pp. 460–471. DOI: 10.1007/978-3-319-53925-6_36.

Hoang, Duc A. and Ryuhei Uehara (2016). "Sliding Tokens on a Cactus". In: *Proceedings of ISAAC 2016*. Ed. by Seok-Hee Hong. Vol. 64. LIPIcs. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 37:1–37:26. DOI: 10.4230/LIPIcs.ISAAC.2016.37.

Ito, Takehiro, Erik D. Demaine, Nicholas J. A. Harvey, Christos H. Papadimitriou, Martha Sideri, Ryuhei Uehara, and Yushi Uno (2011). "On the complexity of reconfiguration problems". In: *Theoretical Computer Science* 412.12, pp. 1054–1065. DOI: 10.1016/j.tcs.2010.12.005.

Kamiński, Marcin, Paul Medvedev, and Martin Milanič (2012). "Complexity of independent set reconfigurability problems". In: *Theoretical Computer Science* 439, pp. 9–15. DOI: 10.1016/j.tcs.2012.03.004.

Bibliography IV

Wrochna, Marcin (2014). "Reconfiguration in bounded bandwidth and treedepth". In: *arXiv preprints*. arXiv: 1405.0847.