Open Problems

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1 Token Sliding Graphs

(Combinatorial) Reconfiguration

studies problems in the following setting:

For any source problem $\mathcal{P}$ (e.g., *Satisfiability*)

- Each *configuration* is a feasible solution of $\mathcal{P}$ (e.g., *satisfying truth assignment*)
  - In general, *the set of all configurations is huge and not part of the input*
- For two configurations A, B, B is *adjacent* to A if it can be obtained from A by applying a given (reconfiguration) rule (e.g., *flipping one bit*) exactly once

Typical questions are:

- **Algorithmic:** *Is there a sequence* of adjacent configurations between two given ones? If so, *is this shortest*? *Can we bound the length* (= the number of times the rule is applied) of such a sequence, if it exists?, and so on
- **Graph-Theoretic:** From the above setting, a *(reconfiguration) graph* can be naturally defined. *Which graph class* does this graph belong to? *Which property* does this graph satisfy? Is this *closed under a graph operation*?, and so on

Token Sliding Graphs: Definition

Token Sliding Graphs ($\mathsf{TS}(G)$ and $\mathsf{TS}_k(G)$)

- **Vertex:** (Size-k) *stable set* of G ($1 \leq k \leq \alpha(G)$)
- **Edge:** Defined via *Token Sliding*

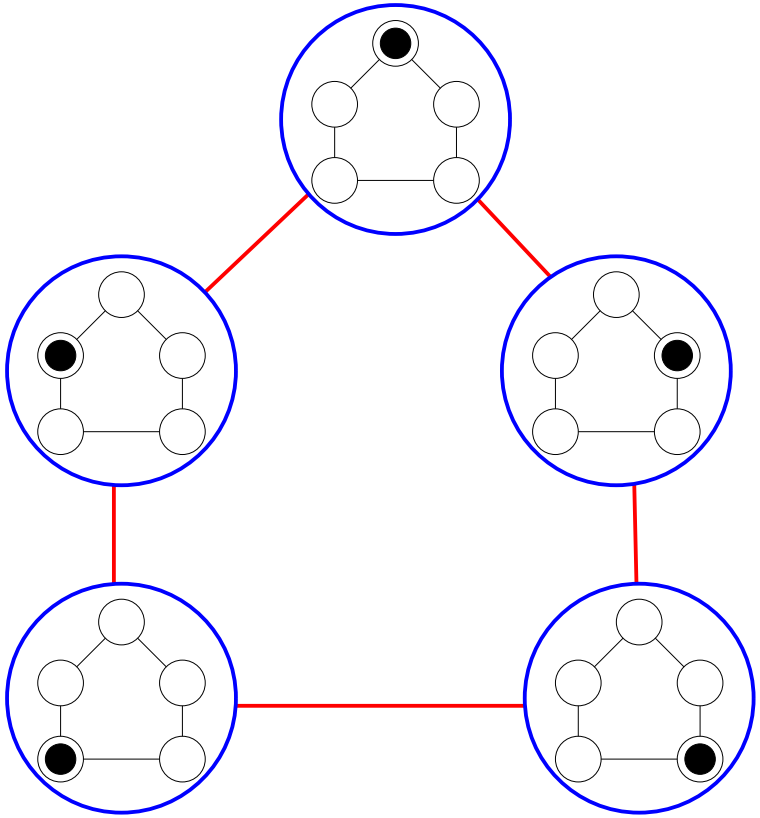

Figure: $\mathsf{TS}_1(C_5)$

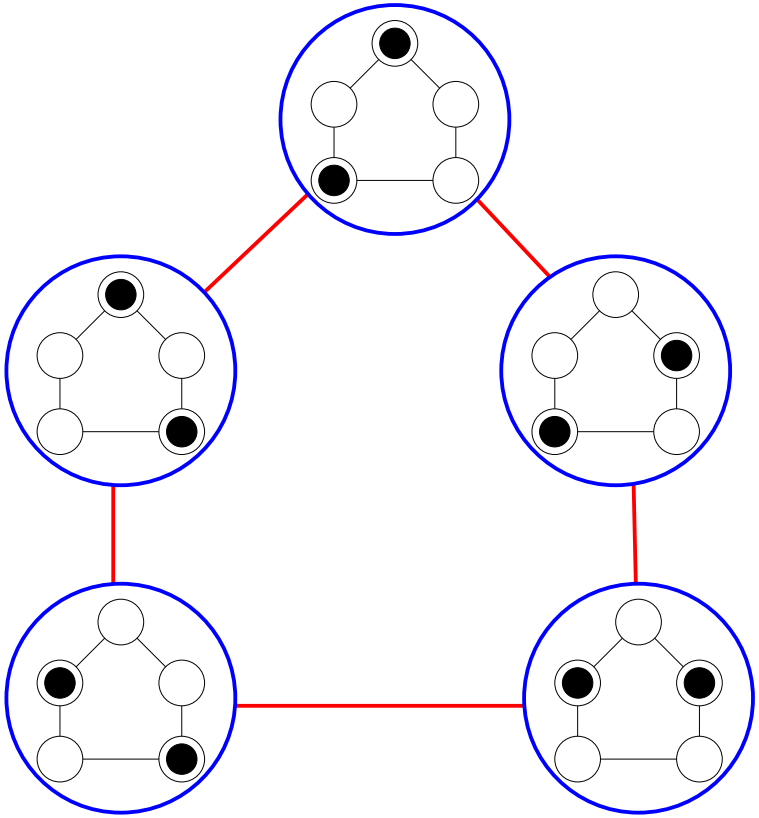

Figure: $\mathsf{TS}_2(C_5)$

Token Sliding Graphs: Motivation

- *Algorithmic questions* have been *well-studied* ([Nishimura 2018], Sections 4 and 5), while *graph-theoretic questions* have *not*
- Similar graphs that have been well-studied in the literature
  - *Fibonacci cube*, first introduced in [Hsu 1993] (see the survey [Klavžar 2013]) [Reconfiguration graph of independent sets on paths under Token Addition/Removal]
  - *Simplex graphs*, first introduced in [Bandelt and De Vel 1989] [Reconfiguration graph of cliques under Token Addition/Removal]
  - *Token graphs*, first studied in [Alavi, Behzad, Erdős, and Lick 1991] (see the survey [Monroy et al. 2012]) [Reconfiguration graph of vertex subsets under Token Sliding]
  - *Matching graphs*, first introduced in [Eroh and Schultz 1998] [Reconfiguration graph of (maximum) matchings under Token Addition/Removal]
  - *Reconfiguration graphs of dominating sets and vertex-colorings*, see the survey [Mynhardt and Nasserasr 2019]

Token Sliding Graphs: Open Problem

We initiate the study of $\mathsf{TS}(G)$ and $\mathsf{TS}_k(G)$ from the graph-theoretic viewpoint

- David Avis and Duc A. Hoang (2023b). "On Reconfiguration Graph of Independent Sets under Token Sliding". In: *Graphs and Combinatorics* 39.3. (article 59). DOI: 10.1007/s00373-023-02644-w. arXiv: 2203.16861
- David Avis and Duc A. Hoang (2023a). "A Note On Acyclic Token Sliding Reconfiguration Graphs of Independent Sets". In: *arXiv preprint*. arXiv: 2301.00317

Some open questions from our research

1. **Given a graph G. What are the necessary and sufficient conditions for G such that $\mathsf{TS}_k(G)$ is a forest?**
   - We partially answer this question in [Avis and Hoang 2023a] when *G is a forest* for $k = 2, 3$ and give a conjecture for $k \geq 4$
   - The case *G is not a forest* requires more insightful ideas
2. **Given a graph G. What are the necessary and sufficient conditions for G to be a TS_k-graph (of some graph H)?**
   - We answer this question in [Avis and Hoang 2023b] for the case G is *a path, a cycle, a complete graph, a complete bipartite graph, or a connected split graph*
   - What about *other graph classes*?

References

- Avis, David and Duc A. Hoang (2023a). "A Note On Acyclic Token Sliding Reconfiguration Graphs of Independent Sets". In: *arXiv preprint*. arXiv: 2301.00317.
- Avis, David and Duc A. Hoang (2023b). "On Reconfiguration Graph of Independent Sets under Token Sliding". In: *Graphs and Combinatorics* 39.3. (article 59). DOI: 10.1007/s00373-023-02644-w. arXiv: 2203.16861.
- Mynhardt, C.M. and S. Nasserasr (2019). "Reconfiguration of Colourings and Dominating Sets in Graphs". In: *50 years of Combinatorics, Graph Theory, and Computing*. Ed. by Fan Chung, Ron Graham, Frederick Hoffman, Ronald C. Mullin, Leslie Hogben, and Douglas B. West. 1st. CRC Press, pp. 171–191. DOI: 10.1201/9780429280092-10.
- Nishimura, Naomi (2018). "Introduction to Reconfiguration". In: *Algorithms* 11.4, p. 52. DOI: 10.3390/a11040052.

References

- Klavžar, Sandi (2013). “Structure of Fibonacci cubes: a survey”. In: *Journal of Combinatorial Optimization* 25.4, pp. 505–522. DOI: 10.1007/s10878-011-9433-z.
- Monroy, Ruy Fabila, David Flores-Peñaloza, Clemens Huemer, Ferran Hurtado, Jorge Urrutia, and David R. Wood (2012). “Token Graphs”. In: *Graphs and Combinatorics* 28.3, pp. 365–380. DOI: 10.1007/s00373-011-1055-9.
- Eroh, Linda and Michelle Schultz (1998). “Matching graphs”. In: *Journal of Graph Theory* 29.2, pp. 73–86. DOI: 10.1002/(SICI)1097-0118(199810)29:2<73::AID-JGT3>3.0.CO;2-9.
- Hsu, Wen-Jing (1993). “Fibonacci cubes-a new interconnection topology”. In: *IEEE Transactions on Parallel and Distributed Systems* 4.1, pp. 3–12. DOI: 10.1109/71.205649.
- Alavi, Yousef, Mehdi Behzad, Paul Erdős, and Don R. Lick (1991). “Double Vertex Graphs”. In: *Journal of Combinatorics, Information & System Sciences* 16.1, pp. 37–50.

References

Bandelt, H-J and Marcel van De Vel (1989). “Embedding topological median algebras in products of dendrons”. In: *Proceedings of the London Mathematical Society* 3.3, pp. 439–453. DOI: 10.1112/plms/s3-58.3.439.