Open Problems

Duc A. Hoang
VNU University of Science, Hanoi, Vietnam
hoanganhduc@hus.edu.vn

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Abstract

This document contains a list of open problems that Duc A. Hoang is interested in. It is also available in HTML format. Please get in touch if you have any idea/comment/question/update. You can find (probably more interesting) open problems from the following resources.

- Open Problem Garden: Graph Theory.
- Open Problems - Graph Theory and Combinatorics, by Douglas B. West.
- List of IWOCA's Open Problems.
- Graph Theory - Favorite Conjectures and Open Problems - 1 (edited by Ralucca Gera, Stephen Hedetniemi, Craig Larson) and Graph Theory - Favorite Conjectures and Open Problems - 2 (edited by Ralucca Gera, Teresa W. Haynes, Stephen T. Hedetniemi).
- Open Problems in Computational Geometry, maintained by Erik D. Demaine, Joseph S. B. Mitchell, Joseph O'Rourke.
- Open Problems in Paramaterized Complexity.
- List of unsolved problems/conjectures in graph theory from Wikipedia.
- Collection of open problems propsed by participants of the Graduate Research Workshop in Combinatorics (GRWC).

Contents

1 Preliminaries

Unless otherwise noted, all graphs are simple, undirected. We often denote by $P_n, C_n, K_n, K_{m,n}$ the *path*, *cycle*, *complete graph*, *complete bipartite graph* on n vertices, respectively. For two graphs G, H, the *disjoint union* of G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. For the terminology and notation not defined here, see the textbooks by Diestel [8] or West [15]. For more details on *Combinatorial Reconfiguration* (https://en.wikipedia.org/wiki/Reconfiguration), see the surveys [5, 12–14] and the wiki page https://reconf.wikidot.com/.

2 Reconfiguration of Independent Sets under Token Sliding

2.1 Definitions

The *double-broom graph* $D_{r,n,s}$ is the tree obtained from a path $P_n = v_1 v_2 \dots v_n$ by attaching r leaves to v_1 and s leaves to v_n, where r, n, s are positive integers. Two independent sets I, J are *adjacent under Token Sliding* (TS) if there exists $u, v \in V(G)$ such that $I \setminus J = \{u\}$, $J \setminus I = \{v\}$, and $uv \in E(G)$. The Token Sliding problem asks whether there is a TS-*sequence* between I and J in G, that is, a sequence of independent sets starting with I and ending with J where any two consecutive members are adjacent under TS. For a fixed positive integer k, the graph $\mathsf{TS}_k(G)$ contains all size-k independent sets as its vertices/nodes, and two nodes are joined by an edge if their corresponding independent sets are adjacent under TS. A graph G is called a TS_k-*graph* if there exists a graph H such that G and $\mathsf{TS}_k(H)$ are isomorphic.

2.2 Conjecture/Question

Open Problem 1. *What is the complexity of Token Sliding on outerplanar graphs? And more generally, on series-parallel graphs (≡ graphs of treewidth at most two)?*

Open Problem 2. *Let G be a graph. What are necessary and sufficient conditions for G such that $\mathsf{TS}_2(G)$ is acyclic?*

Conjecture 3. *Let G be a forest. Then $\mathsf{TS}_k(G)$ is a forest if and only if G is $\{2K_2 + (k-2)K_1, D_{2,2,2} + (k-2)K_1, D_{2,4,2} + (k-3)K_1\}$-free, for some integer $k \geq 4$.*

2.3 Background/Motivation

Independent Set Reconfiguration (ISR) is one of the most well-studied problems in the Combinatorial Reconfiguration [5, 13]. Among several variants of ISR, Token Sliding is of particular interest. One of the main reasons is that in Token Sliding one often needs to deal with the situation where a token must make "detours" by "moving away" (and then moving back later) to allow some other token to move [7]. Alternatively, this can be seen as a kind of "bottleneck effect" [3] where a token may not be able to "reach" some vertices which are "far apart" from all tokens because some tokens "block the way".

Our motivation for Open Problem 1 comes from the following two results that both appeared in 2014.

Theorem 4 ([16])**.** *There exists a constant c such that Token Sliding (and two other variants of ISR) is* PSPACE-*complete on graphs of bandwidth at most c.*

Indeed, since a graph of bandwidth at most c also has pathwidth and treewidth at most c, Theorem 4 holds for graphs of pathwidth/treewidth at most c. Moreover, for $c = 1$,

Theorem 5 ([7])**.** *Token Sliding is in* P *on trees.*

Naturally, on graphs of treewidth at most c, one may ask what the value of c that separates "hard" from "easy" is. Toward answering this question, a first step is probably to consider $c = 2$.

Our motivation for Open Problem 2 and Conjecture 3 comes from [1, 2]. In [1], the authors initiated the study of $\mathsf{TS}_k(G)$ from a purely graph-theoretic viewpoint. They continued their study in [2] focusing on those that are acyclic. Open Problem 2 involves characterizing general graphs, which seems to be quite challenging and a first step may be to consider well-known graph classes. Toward this direction, in [2], the authors proved a forbidden induced subgraph characterization of a tree G satisfying that $\mathsf{TS}_k(G)$ is acyclic where $k \in \{2, 3\}$ and Conjecture 3, if true, will provide a complete analysis of which trees having acyclic TS_k-graphs.

3 Reconfiguring k-Path Vertex Covers on Trees under Token Sliding

3.1 Definitions

For a fixed integer $k \geq 2$, a k-*path vertex cover* of a graph G is a vertex subset I such that any path on k vertices contains at least one member from I. Two k-path vertex covers are *adjacent under Token Sliding (*TS*)* if there exists $u, v \in V(G)$ such that $I \setminus J = \{u\}$, $J \setminus I = \{v\}$, and $uv \in E(G)$. The k-Path Vertex Cover Reconfiguration under Token Sliding (k-PVCR-TS) problem asks if there is a TS-*sequence* between two given k-path vertex covers I and J in G, that is, a sequence of k-path vertex covers starting with I and ending with J where any two consecutive members are adjacent under TS.

3.2 Conjecture/Question

Open Problem 6. *What is the complexity of k-PVCR-TS on bipartite graphs? Or more restrictedly, on trees?*

3.3 Background/Motivation

The k-PVCR-TS problem was first introduced in [11]. The complexities of k-PVCR-TS and some other variants have been characterized in [11] for some graph classes, including planar and bounded bandwidth graphs, trees, paths, and cycles. However, k-PVCR-TS on trees remains open. Our motivation comes from an attempt in [10] to solve this problem for a subclass of trees called *caterpillars* (i.e., trees obtained by attaching leaves to a central path). Unlike the cases for independent sets, a token may have to make "detours" by "moving closer" (and then moving back later) to allow some other tokens to move.

4 Recoloring under Token Swapping

4.1 Definitions

For a fixed integer $k \geq 1$, a *proper k-coloring* α of a graph G is a function $\alpha : V(G) \to \{1, 2, \dots, k\}$ such that for any edge $uv \in E(G)$ we have $\alpha(u) \neq \alpha(v)$. If G has a proper k-coloring, we say that it is *k-colorable*. Two proper k-colorings α, β are *adjacent under Token Swapping* if there exist $u, v \in V(G)$ such that $\alpha(u) = \beta(v)$, $\alpha(v) = \beta(u)$, $\alpha(w) = \beta(w)$ for any $w \in V(G) - \{u, v\}$, and $uv \in E(G)$. The k-Recoloring under Token Swapping (k-RTS) problem asks if there is a sequence of adjacent proper k-colorings of G under Token Swapping between two given proper k-colorings α, β. The graph $C_k^{RTS}(G)$ takes all proper k-colorings of G as its nodes and their adjacency are defined under Token Swapping.

4.2 Conjecture/Question

Open Problem 7. *What is the complexity of k-RTS on trees for some fixed $k \geq 3$?*

Open Problem 8. *What is the smallest value of k such that $C_k^{RTS}(G)$ is connected for some given graph G?*

Open Problem 9. *What are necessary and sufficient conditions for a k-colorable graph G such that $C_{k+1}^{RTS}(G)$ is connected?*

4.3 Background/Motivation

The k-RTS problem lies between two of the most well-studied problems in Combinatorial Reconfiguration: Token Swapping [17] and Vertex Recoloring [4]. (See [12, 13] for more details.) RTS was first proposed in [9] where the authors claimed that RTS remains PSPACE-complete on planar graphs and can be solved in polynomial time on paths ($k = 3$) and cographs (any k). Up to present, their results have not yet been published. The goal of our proposed open problems is to obtain a better understanding on the similarities and differences between $C_k^{RTS}(G)$ and the well-stuided graph $C_k(G)$—the graph whose nodes are proper k-colorings of G and two nodes are adjacent if one can be obtained from the other by recoloring exactly one vertex.

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