Moving Tokens on Graphs

Some Known Results and Open Questions

Hoang Anh Duc
May 28, 2020

hoanganhduc@ces.kyutech.ac.jp
Research Seminar at Kyutech Algorithms Group
In this talk, I will

1. Briefly introduce reconfiguration problems.
2. Explain some known results on
   - Independent Set Reconfiguration
   - $k$-Path Vertex Cover Reconfiguration
   as examples about specific reconfiguration problems.
3. Introduce some relevant open questions.
Reconfiguration Problems
Reconfiguration Problems

- **Instance:**
  - Two configurations $A$, $B$.
    - E.g., two states of Rubik’s Cube Puzzle.
  - A reconfiguration rule (defining whether two arbitrary configurations are adjacent).
    - E.g., rotating one face of the cube 90, 180, or 270 degrees.

- **Question:** Does there exist a sequence of adjacent configurations that transforms $A$ into $B$?

![Figure 1: An instance of Rubik’s Cube Puzzle.](image)
Reconfiguration Problems

Figure 2: Reconfiguration. (© Anna Lubiw, in her tutorial “Reconfiguration and geometry” at CoRe2019.)
Naturally, we can define the corresponding reconfiguration graph.

- Each configuration is a node.
- Two nodes (configurations) $X, Y$ are adjacent if one can be obtained from the other by applying the given reconfiguration rule exactly once.
Reconfiguration Problems

Naturally, we can define the corresponding reconfiguration graph.

- Each configuration is a node.
- Two nodes (configurations) $X, Y$ are adjacent if one can be obtained from the other by applying the given reconfiguration rule exactly once.

One may ask

- **Reachability**: a path between two nodes of a reconfiguration graph?
- **Shortest Reconfiguration**: find a shortest path (if exists) between two nodes of a reconfiguration graph?
- **Connectivity**: a reconfiguration graph is connected?
- **Diameter**: the diameter of a reconfiguration graph is bounded?
Independent Set Reconfiguration
An independent set of a graph $G$ is a vertex-subset $I \subseteq V(G)$ such that no two members of $I$ are connected by an edge in $G$.

**Figure 3:** Examples of independent sets.
Imagine that each member of an independent set contains a token.

Figure 4: Token Sliding: move a token to one of its neighbors.
Imagine that each member of an independent set contains a token.

**Figure 5:** Token Jumping: move a token to one of unoccupied vertices.

**Remark:** Token Jumping generalizes Token Sliding.
Token Addition and Removal (TAR($u$))

Imagine that each member of an independent set contains a token.

Figure 6: Token Addition and Removal: add or remove a token such that the number of tokens is always at least $u$.

Remark: Token Addition and Removal generalizes Token Jumping.
Independent Set Reconfiguration (ISR)

- **Instance:**
  - Two independent sets $I, J$ of a graph $G$.
  - One of the following reconfiguration rules: TS, TJ, TAR($u$).

- **Question:** Does there exist a sequence of adjacent independent sets that transforms $I$ into $J$?

**Figure 7:** Examples of ISR’s yes-instances.
Independent Set Reconfiguration (ISR)

- ISR is PSPACE-complete on several well-known graph classes, including
  - perfect graphs [Kamiński, Medvedev, and Milanič 2012],
  - planar graphs of maximum degree 3 [Hearn and Demaine 2005]
  - bounded treewidth graphs [Wrochna 2018],

Naturally, one may ask: which graph class makes the problem “easy/hard”? 

Open Question: which parameter makes the problem “easy/hard”?

A natural parameter: the number of tokens. [this talk]
Independent Set Reconfiguration (ISR)

- ISR is PSPACE-complete on several well-known graph classes, including:
  - perfect graphs [Kamiński, Medvedev, and Milanič 2012],
  - planar graphs of maximum degree 3 [Hearn and Demaine 2005],
  - bounded treewidth graphs [Wrochna 2018],

- Naturally, one may ask:
  - which graph class makes the problem “easy/hard”?  
    **Open Question:** graphs of treewidth $\leq 2$?
  - which parameter makes the problem “easy/hard”?  
    A natural parameter: the number of tokens.  
    [this talk]
ISR Parameterized by the Number of Tokens

- **Instance:**
  - Two independent sets $I, J$ of a graph $G$.
  - One of the following reconfiguration rules: TS, TJ, TAR($u$).

- **Parameter:** The number of tokens $k$.

- **Question:** Does there exist a sequence of adjacent independent sets that transforms $I$ into $J$?

This problem is $\overline{\text{W}}[1]$-hard

- under TAR [Mouawad et al. 2017],
- under TJ [Ito et al. 2020]
- under TS [this talk]

and is in FPT

- on planar graphs under TJ [Ito, Kamiński, and Ono 2014].
**Remind: How to prove \( W[1] \)-hardness?**

**Parameterized Reduction from problem \( P \) to problem \( Q \)**

is a function \( \phi \) with the following properties:

- \( \phi(x) \) can be computed in time \( f(k) \cdot |x|^{O(1)} \), where \( k \) is the parameter of \( x \),
- \( \phi(x) \) is a yes-instance of \( Q \) \( \iff \) \( x \) is a yes-instance of \( P \),
- If \( k \) is the parameter of \( x \) and \( k' \) is a parameter of \( \phi(x) \), then \( k' \leq g(k) \) for some function \( g \).

- Transforming an **Independent Set** instance \((G, k)\) into a **Vertex Cover** instance \((G, n - k)\) is **not** a parameterized reduction.
- Transforming an **Independent Set** instance \((G, k)\) into a **Clique** instance \((\overline{G}, k)\) is a parameterized reduction.
ISR Parameterized by the Number of Tokens under TS

**Claim:** $G'$ has an independent set of size $\geq k'$ if and only if $I$ can be reconfigured into $J$ under TS in $G$.

**Open Question:** FPT algorithm on planar graphs?
$k$-Path Vertex Cover
Reconfiguration
$k$-Path Vertex Cover ($k$-PVC)

A $k$-path vertex cover of a graph $G$ is a vertex-subset $I \subseteq V(G)$ such that each path on $k$ vertices contains a member of $I$.

Figure 9: Examples of 3-path vertex covers.

Remind: When $k = 2$, it is called a vertex cover.
\(k\)-Path Vertex Cover Reconfiguration (\(k\)-PVCR)

- **Instance:**
  - Two \(k\)-path vertex covers \(I, J\) of a graph \(G\).
  - One of the following reconfiguration rules: TS, TJ, TAR\((u)\).

- **Question:** Does there exist a sequence of adjacent \(k\)-path vertex covers that transforms \(I\) into \(J\)?

- **Remark:** Unlike in ISR, the TAR\((u)\) rule requires that the number of tokens must be at most \(u\).

**Theorem (Hoang, Suzuki, and Yagita 2020)**
\(k\)-PVCR is \(\text{PSPACE}\)-complete on planar graphs of maximum degree 3, bounded treewidth graphs (under all rules), and bipartite graphs and chordal graphs (under TS). On the other hand, it can be solved in polynomial time on trees under TJ and TAR.

**Open Question:** \(k\)-PVCR under TS on trees?
Instance:
- Two $k$-path vertex covers $I, J$ of a graph $G$.
- One of the following reconfiguration rules: TS, TJ, TAR($u$).

Parameter: The number of tokens $t$.

Question: Does there exist a sequence of adjacent $k$-path vertex covers that transforms $I$ into $J$?
$k$-PVCR Parameterized by The Number of Tokens

- **Instance:**
  - Two $k$-path vertex covers $I, J$ of a graph $G$.
  - One of the following reconfiguration rules: TS, TJ, TAR($u$).
- **Parameter:** The number of tokens $t$.
- **Question:** Does there exist a sequence of adjacent $k$-path vertex covers that transforms $I$ into $J$?

**Observation:** From this meta-theorem, $k$-PVCR parameterized by the number of tokens is in FPT.

**Theorem (Mouawad et al. 2017)**

If a $t$-subset problem $Q$ is superset-closed and has an FPT algorithm to enumerate all its minimal solutions of cardinality at most $t$, the number of which is bounded by a function of $t$, then $Q$ **reconfiguration parameterized by $t$ is in FPT, as well as the search and shortest path variants.**

**Open Question:** More efficient FPT algorithms where $k$ is small?
Resources and References
# Resources and References

## Surveys on Reconfiguration


## Open Problems from CoRe 2019


