

10. Bessel Functions of Fractional Order

Mathematical Properties

10.1. Spherical Bessel Functions

Definitions

Differential Equation

10.1.1

$$z^2 w'' + 2zw' + [z^2 - n(n+1)]w = 0 \quad (n=0, \pm 1, \pm 2, \dots)$$

Particular solutions are the *Spherical Bessel functions of the first kind*

$$j_n(z) = \sqrt{\frac{1}{2}\pi/z} J_{n+\frac{1}{2}}(z),$$

the *Spherical Bessel functions of the second kind*

$$y_n(z) = \sqrt{\frac{1}{2}\pi/z} Y_{n+\frac{1}{2}}(z),$$

and the *Spherical Bessel functions of the third kind*

$$h_n^{(1)}(z) = j_n(z) + iy_n(z) = \sqrt{\frac{1}{2}\pi/z} H_{n+\frac{1}{2}}^{(1)}(z),$$

$$h_n^{(2)}(z) = j_n(z) - iy_n(z) = \sqrt{\frac{1}{2}\pi/z} H_{n+\frac{1}{2}}^{(2)}(z).$$

The pairs $j_n(z)$, $y_n(z)$ and $h_n^{(1)}(z)$, $h_n^{(2)}(z)$ are linearly independent solutions for every n . For general properties see the remarks after 9.1.1.

Ascending Series (See 9.1.2, 9.1.10)

10.1.2

$$j_n(z) = \frac{z^n}{1 \cdot 3 \cdot 5 \dots (2n+1)} \left\{ 1 - \frac{\frac{1}{2}z^2}{1!(2n+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)} - \dots \right\}$$

10.1.3

$$y_n(z) = -\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{z^{n+1}} \left\{ 1 - \frac{\frac{1}{2}z^2}{1!(1-2n)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)} - \dots \right\} \quad (n=0, 1, 2, \dots)$$

Limiting Values as $z \rightarrow 0$

$$10.1.4 \quad z^{-n} j_n(z) \rightarrow \frac{1}{1 \cdot 3 \cdot 5 \dots (2n+1)}$$

10.1.5

$$z^{n+1} y_n(z) \rightarrow -1 \cdot 3 \cdot 5 \dots (2n-1) \quad (n=0, 1, 2, \dots)$$

Wronskians

$$10.1.6 \quad W\{j_n(z), y_n(z)\} = z^{-2}$$

10.1.7

$$W\{h_n^{(1)}(z), h_n^{(2)}(z)\} = -2iz^{-2} \quad (n=0, 1, 2, \dots)$$

Representations by Elementary Functions

10.1.8

$$j_n(z) = z^{-1} [P(n+\frac{1}{2}, z) \sin(z - \frac{1}{2}n\pi) + Q(n+\frac{1}{2}, z) \cos(z - \frac{1}{2}n\pi)]$$

10.1.9

$$y_n(z) = (-1)^{n+1} z^{-1} [P(n+\frac{1}{2}, z) \cos(z + \frac{1}{2}n\pi) - Q(n+\frac{1}{2}, z) \sin(z + \frac{1}{2}n\pi)]$$

$$P(n+\frac{1}{2}, z) = 1 - \frac{(n+2)!}{2! \Gamma(n-1)} (2z)^{-2} + \frac{(n+4)!}{4! \Gamma(n-3)} (2z)^{-4} - \dots$$

$$= \sum_0^{[n]} (-1)^k (n+\frac{1}{2}, 2k) (2z)^{-2k}$$

$$Q(n+\frac{1}{2}, z) = \frac{(n+1)!}{1! \Gamma(n)} (2z)^{-1} - \frac{(n+3)!}{3! \Gamma(n-2)} (2z)^{-3} + \frac{(n+5)!}{5! \Gamma(n-4)} (2z)^{-5} - \dots$$

$$= \sum_0^{[(n-1)]} (-1)^k (n+\frac{1}{2}, 2k+1) (2z)^{-2k-1}$$

$(n=0, 1, 2, \dots)$

$$(n+\frac{1}{2}, k) = \frac{(n+k)!}{k! \Gamma(n-k+1)}$$

$n \backslash k$	1	2	3	4	5
1	2				
2	6	12			
3	12	60	120		
4	20	180	840	1680	
5	30	420	3360	15120	30240

10.1.10

$$j_n(z) = f_n(z) \sin z + (-1)^{n+1} f_{-n-1}(z) \cos z$$

$$f_0(z) = z^{-1}, \quad f_1(z) = z^{-2}$$

$$f_{n-1}(z) + f_{n+1}(z) = (2n+1)z^{-1}f_n(z) \\ (n=0, \pm 1, \pm 2, \dots)$$

The Functions $j_n(z)$, $y_n(z)$ for $n=0, 1, 2$

$$10.1.11 \quad j_0(z) = \frac{\sin z}{z}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z$$

10.1.12

$$y_0(z) = -j_{-1}(z) = -\frac{\cos z}{z}$$

$$y_1(z) = j_{-2}(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

$$y_2(z) = -j_{-3}(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right) \cos z - \frac{3}{z^2} \sin z$$

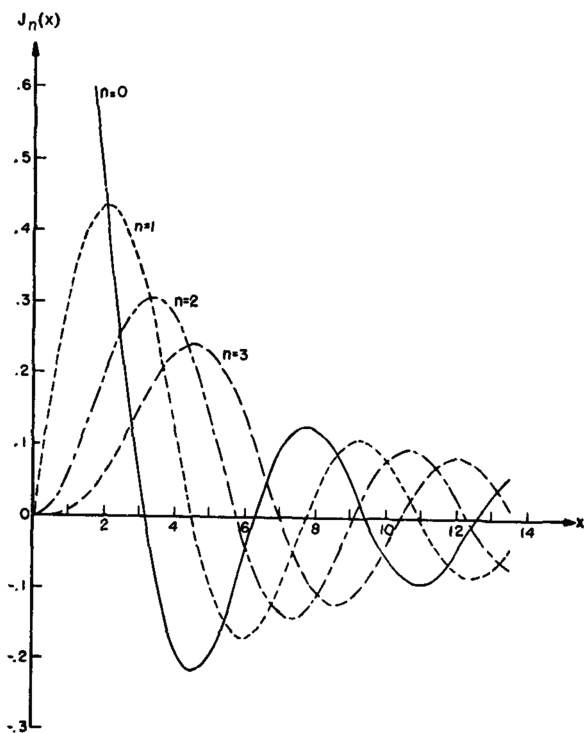


FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

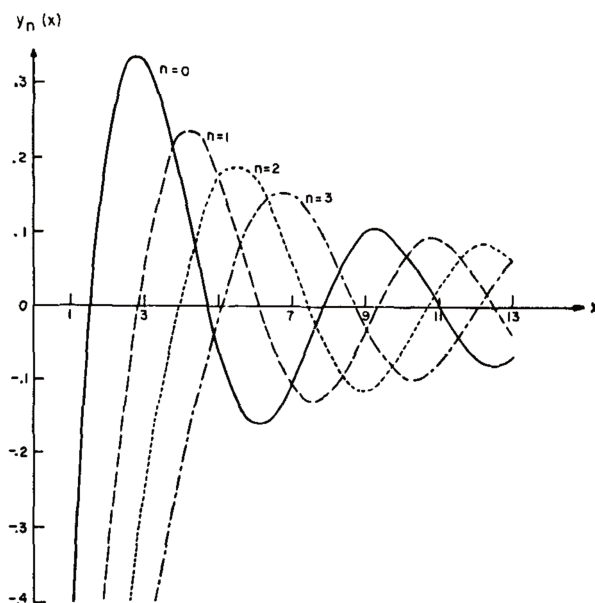


FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

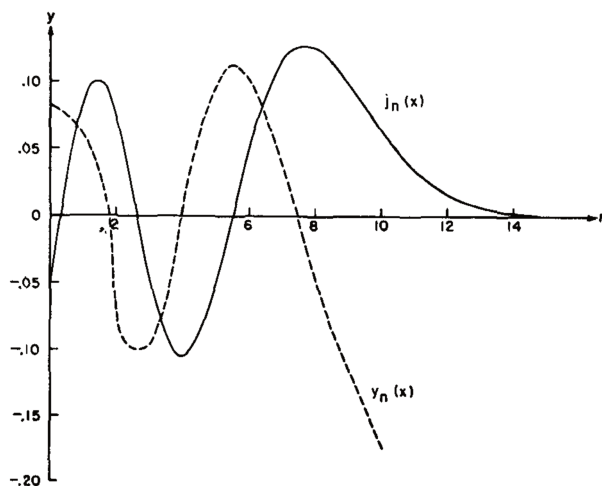


FIGURE 10.3. $j_n(x)$, $y_n(x)$. $x=10$.

Poisson's Integral and Gegenbauer's Generalization

$$10.1.13 \quad j_n(z) = \frac{z^n}{2^{n+1}n!} \int_0^\pi \cos(z \cos \theta) \sin^{2n+1} \theta \, d\theta \\ (\text{See 9.1.20.})$$

10.1.14

$$= \frac{1}{2} (-i)^n \int_0^\pi e^{iz \cos \theta} P_n(\cos \theta) \sin \theta \, d\theta \\ (n=0, 1, 2, \dots)$$